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IN

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SOLUTIONS OF EXAMPLES

IN

ELEMENTARY HYDROSTATICS

BY

William Henry
W. H. BESANT, Sc.D., F.R.S.
FELLOW OF ST JOHN'S COLLEGE.

SECOND EDITION. REVISED.

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1893

PREFACE TO THE FIRST EDITION.

I HAVE been frequently asked to produce solutions of the examples in my Treatise on Elementary Hydrostatics, but the pressure of other work has prevented me from undertaking the task of preparing them.

These solutions have been almost entirely drawn up by Mr A. W. FLUX, Fellow of St John's College, and I am much indebted to him for the labour which he has bestowed upon the work.

I hope that they will be found to be useful and helpful, both to teachers and to students.

No figures have been given, but the student will find no serious difficulty in drawing figures for himself when necessary, and he will find it greatly to his advantage to do so.

W. H. BESANT.

January 1891.

PREFACE TO THE SECOND EDITION.

IN the latest edition of the Treatise on Elementary Hydrostatics, the fifteenth, considerable changes were made in the text, and much additional matter was inserted.

The examples and problems were also rearranged, some useless examples were removed, and a number of new examples, taken chiefly from recent examination papers, were added to the various groups.

The present edition of the Solutions has been carefully arranged so as to be in complete accordance with the fifteenth edition of the Treatise.

I have again to thank Mr A. W. FLUX for valuable assistance in the writing out of Solutions and in the revision of proof sheets.

W. H. BESANT.

February 1893.

TABLE OF CONTENTS.

ERRATA.

PAGE

2. Third line from the end, *for* on sq. inches *read* on π sq. inches.

4. Line 12, *read* 16×1728.

6. Lines 12, 13, 14; the factor 62·5 is omitted.

13. Line 19, *read* $w \cdot \frac{\beta}{3} \cdot \frac{\beta^2 h}{2}$.

17. „ 2, *for* lengths *read* depths.

21. Ex. 14, line 3, *read* $\frac{4}{3}\pi r'^3 \rho$.

29. Ex. 46. This should run as follows.

The shifting of 20 tons is a gain on one side of the moment of 40 tons;

$$\therefore \quad 9000\, h \sin\theta = 40 \times 21 \cos\theta,$$

also

$$\tan\theta = \frac{10}{20 \times 12} = \frac{1}{24}$$

$$\therefore \quad h = \frac{84}{900} \cot\theta = 2{\cdot}24.$$

46. Ex. 8. In order to solve this question, another condition must be given.

Taking the general case, suppose that a common hydrometer has a portion of its bulb chipped off, and that, when placed in liquids of densities α, β and γ, it indicates densities α', β', and γ' respectively.

Let V be volume of instrument, ρ its density, U the volume chipped off, κ the cross section of the stem.

Then if x, y, z are the lengths of stem above the surface,

$$\rho V = \alpha'(V - \kappa x), \quad \rho(V - U) = \alpha(V - U - \kappa x),$$

$$\rho V = \beta'(V - \kappa y), \quad \rho(V - U) = \beta(V - U - \kappa y),$$

$$\rho V = \gamma'(V - \kappa z), \quad \rho(V - U) = \gamma(V - U - \kappa z).$$

$$\therefore \quad \frac{1}{\alpha'} - \frac{1}{\alpha} + \frac{1}{\alpha}\frac{U}{V} = \frac{1}{\beta'} - \frac{1}{\beta} + \frac{1}{\beta}\frac{U}{V} = \frac{1}{\gamma'} - \frac{1}{\gamma} + \frac{1}{\gamma} \cdot \frac{U}{V},$$

and

$$\frac{\gamma}{\gamma'} = \frac{\alpha\beta(\beta' - \alpha')}{\alpha'\beta'(\beta - \alpha) + \gamma'(\alpha\beta' - \alpha'\beta)}.$$

If α, β are known, and if α', β', γ' are observed, this determines γ. *Tripos Examination*, 1890.

ELEMENTARY HYDROSTATICS.

SOLUTIONS OF EXAMPLES.

CHAPTER I.

EXAMINATION.

4. (1) THE unit of area is a square inch, the pressure on which is $10\frac{1}{2}$ pounds weight.

(2) The unit of area is four square inches, the pressure on which is 42 pounds weight.

5. Suppose h so small that the pressure may be considered uniform over the rectangle, the area of which is bh.

This uniform pressure is

$$wbh(a+h)/bh = w(a+h) = wa$$

when h vanishes, i.e. at any point in the upper side.

6. Area of larger pipe $= 9 \times$ area of smaller pipe;

$\therefore$ required force $= 9 \times 20 = 180$ lbs. weight.

8. Area of $B = 64 \times 36 \times 36 \times$ area of A;

$\therefore$ mass supported by $B = 64 \times 36 \times 36$ lbs. $= 37\frac{1}{35}$ tons.

9. If a be the length of the side of the square, $2b$ the other side of the rectangular lid, W its weight, p the pressure when the lid is on the point of lifting,

Taking moments about the line of the hinge

$$pa^2 \times a = W \times b \text{ or } p = \frac{Wb}{a^3}.$$

10. Here $\qquad a = \frac{1}{4},\ b = \frac{5}{4}.$

$$W = \frac{p \cdot a^3}{b} = \frac{800}{5 \times 16} = 10 \text{ lbs. wt.}$$

11. The distance through which the larger piston moves is one-ninth of that moved through by the piston in the smaller pipe.

12. Let the base of the triangle be $2n$ times its height and let h be the height.

$$\text{The area} = nh^2.$$

$$\text{The length of } PQ = 2nx.$$

$$\therefore \text{ Area of } APQ = nx^2.$$

$$\therefore \text{ Mean pressure on } APQ = p/n.$$

Since this is independent of x the pressure is uniform over the whole triangle.

13. Pressure on a circular portion having its centre at the fixed point, and of radius $r = \lambda r^3$.

If the radius be $r+x$ the pressure is $\lambda\,(r+x)^3$.

The increase in area is $\pi\{(r+x)^2 - r^2\} = \pi\,(2rx+x^2)$.

$\therefore$ Mean pressure over the area added

$$= \frac{\lambda}{\pi}\,\frac{(r+x)^3 - r^3}{2rx+x^2} = \frac{\lambda}{\pi}\,\frac{3r^2+3rx+x^2}{2r+x}.$$

$$= \frac{3\lambda}{2\pi}\,r$$

when x is diminished indefinitely.

And this is the pressure *at* any point on the circumference of the circular portion of radius r.

14. The pressure is $\dfrac{10000}{14\frac{1}{2}\times 36\pi}$ atmospheres.

$\therefore$ the compression $= \dfrac{2\times 10000}{29\times 36\pi}\times \cdot 00005$ of the whole volume;

$\therefore$ distance through which piston is compressed

$$= \frac{10\times 12}{29\times 36\pi} = \frac{1}{27\cdot 3}\text{ inch.}$$

15. Total volume of water in the two tubes $= \frac{185}{72}\pi$ cubic feet.

Let x inches be the distance through which the piston is forced.

$$\text{Compression} = \frac{\pi x}{144\times 12}\Big/\frac{185}{72}\pi = \frac{x}{4440}.$$

This is produced by a pressure of $58\,\pi$ lbs. wt. on π sq. inches, which is about 4 atmospheres.

$$\therefore\ 4\times\cdot 000049 = \frac{x}{4440}\,1 \text{ or } x = \cdot 87024 \text{ inches.}$$

CHAPTER II.

EXAMINATION.

2. MASS of a cubic foot of water $= 62{\cdot}5$ lbs.

$\therefore$ Mass of a cubic yard $= 1687{\cdot}5$ lbs.

$$\text{Mass of a cubic inch} = \frac{62{\cdot}5}{1728}$$

$$= {\cdot}036168\dot{9}81\dot{4} \text{ lbs.}$$

Mass of a cubic foot of mercury $= 848$ lbs.

$\therefore$ Mass of a cubic yard $= 22896$ lbs.

Mass of a cubic inch $= {\cdot}49\dot{0}7\dot{4}$ lbs.

3. 1 cub. foot $= 28316$ cub. cm.

$\therefore$ Mass of 1 cub. foot of water $= 28316$ grammes.

Mass of 1 cub. foot of mercury

$= 28316 \times 13{\cdot}568 = 384191{\cdot}488$ grammes.

4. Mass of 1 cub. cm. of water $= 1$ gramme

$= {\cdot}0022046$ lbs. $= {\cdot}0352736$ ounces

or $= \dfrac{1}{28{\cdot}3495}$ ounces. (See Art. 22.)

5. Mass of 1 cub. foot $= 2831{\cdot}6$ grammes

$= 2831{\cdot}6 \times {\cdot}0022046$ lbs.

$= 6{\cdot}2425$ lbs.

6. A cubic yard of cork weighs as much as

${\cdot}24$ cub. yds. $= 6{\cdot}48$ cub. feet of water.

7. Required sp. gr. $= \dfrac{11 \times 19{\cdot}4 + 8{\cdot}84}{12} = 18{\cdot}52.$

8. Mixture of equal volumes has sp. gr. $\frac{1}{2}(5+7)=6$.

Mixture of equal weights has sp. gr. $\dfrac{2}{\frac{1}{5}+\frac{1}{7}}$ or $5\frac{5}{6}$.

9. The required weight

$$=27\times1728\times{\cdot}45\times5=104976 \text{ lbs. wt.}$$

10. $2V$, V being the volumes of the two fluids which are mixed, $2V$ is the volume of the mixture.

$$\therefore \text{ density of mixture}=\frac{2V\rho+2V\rho}{2V}=2\rho.$$

11. The required weight

$$=27\times{\cdot}12\times1000 \text{ ozs.}=202\tfrac{1}{2} \text{ lbs. wt.}$$

12. A cubic inch of water weighs $\dfrac{1000}{1728\times16}$ lbs.

$\therefore$ the sp. gr. of the substance

$$=\frac{1625}{3456}\times\frac{16\overset{\times}{+}1728}{1000}=13.$$

13. s, s', s'', σ being the densities of the three fluids and of the mixture,

$$s+s'+s''=3\sigma; \qquad \therefore\ s=3\sigma-s'-s''.$$

14. Let the resulting volume be $V+V'-v$.

Then $$s(V+V'-v)=V\sigma+V'\sigma';$$

$$\therefore\ v=\frac{V(s-\sigma)+V'(s'-\sigma')}{s}.$$

Examples.

1. Let V, σ be the volume and density of one fluid, mV, $n\sigma$ the volume and density of the other.

$$\rho(m+1)V=V\sigma+mnV\sigma,$$

$$\therefore\ \sigma=\rho\frac{1+m}{1+mn}.$$

2. Let $2V$ be the volume of each fluid.

The volume of the mixture is $3V$. Its density being σ

$$3V\sigma=2V\rho+4V\rho; \qquad \therefore\ \sigma=2\rho.$$

3. The densities of the n fluids being

$$\rho, 2\rho, 3\rho, \ldots\ldots n\rho$$

when equal volumes are mixed, the density of the mixture is

$$\frac{1+2+3+\ldots+n}{n}\rho = \frac{n+1}{2}\rho.$$

When the volumes are in the ratios of $1.2.3\ldots n$, the density of the mixture

$$= \frac{1^2+2^2+\ldots+n^2}{1+2+\ldots+n}\rho = \frac{2n+1}{3}\rho.$$

When the volumes are in the ratios of

$$n.\overline{n-1}\ldots 3.2.1,$$

the density of the mixture

$$= \frac{n+2(n-1)+3(n-2)+\ldots+n}{n+\overline{n-1}+\ldots+2+1}\rho = \frac{n+2}{3}\rho.$$

4. s_1, s_2 being the sp. gr. of the two fluids,

$$\sigma = \tfrac{1}{2}(s_1+s_2), \qquad \sigma' = \tfrac{1}{3}(s_1+2s_2);$$

$$\therefore\ s_1 = 4\sigma - 3\sigma', \qquad s_2 = 3\sigma' - 2\sigma.$$

5. Let s_1, s_2 be the specific gravities of the fluids,

$$\frac{s_1+s_2}{2} = \frac{4}{3} \cdot \frac{2}{\frac{1}{s_1}+\frac{1}{s_2}};$$

$$\therefore\ 3s_1^2 - 10s_1s_2 + 3s_2^2 = 0;$$

$$\therefore\ s_1 : s_2 = 1 : 3 \text{ or } 3 : 1.$$

6. Let x gallons be the quantity required,

$$\frac{x+10{\cdot}31}{x+10} = 1{\cdot}021; \ \therefore\ x = \frac{1}{{\cdot}21} = 4{\cdot}7619\ldots$$

7. Volume of earth

$$= \tfrac{4}{3}\pi\,(1{\cdot}275)^3 . 10^{27} \text{ cubic centimeters.}$$

$$\therefore\ \text{mass of earth} = \tfrac{4}{3}\pi\,(1{\cdot}275)^3 \times 5{\cdot}67 \times 10^{27} \text{ grammes}$$
$$= 6{\cdot}15 \times 10^{27} \text{ grammes about.}$$

8. Let l feet be the unit of length.

The mass of l^3 cubic feet of water is

$$1000\,l^3 \text{ ozs.} = \frac{7000000}{16}\,l^3 \text{ grains.}$$

$\therefore$ the weight of a unit of volume of the substance is the weight of

$$\frac{7000000}{16} l^3 s \text{ grains,}$$

if s denote its specific gravity.

Now, this weight is s units, i.e. $28s$ grains wt.

$$\therefore\ 28 = \frac{7000000}{16} l^3$$

or

$$l^3 = \frac{64}{1000000}$$

and

$$l = \frac{4}{100} = \frac{1}{25} \text{ foot.}$$

9. Let s, σ be the sp. gr. of A and B.

V the number of cub. feet in a gallon.

$$Vs \times 62{\cdot}5 + \lambda_1 = \left(V + \frac{\lambda_1}{62{\cdot}5\sigma}\right)\sigma_1 \times 62{\cdot}5$$

$$= V\sigma_1 \times 62{\cdot}5 + \frac{\lambda_1 \sigma_1}{\sigma}$$

$$\therefore\ V(s - \sigma_1) = \lambda_1 . \frac{\sigma_1 - \sigma}{\sigma}.$$ × 62·5

So also

$$V(s - \sigma_2) = \lambda_2 . \frac{\sigma_2 - \sigma}{\sigma}$$ × 62·5

$$V(s - \sigma_3) = \lambda_3 . \frac{\sigma_3 - \sigma}{\sigma}$$ × 62·5

$$\therefore\ \frac{\lambda_1(\sigma_1 - \sigma)}{s - \sigma_1} = \frac{\lambda_2(\sigma_2 - \sigma)}{s - \sigma_2} = \frac{\lambda_3(\sigma_3 - \sigma)}{s - \sigma_3}$$

which equations determine s and σ.

10. Let s, σ be the specific gravities.

(1) Weights proportional to $\frac{1}{s}$, $\frac{1}{\sigma}$ are mixed.

$$\text{Sp. gr. of mixture} = \frac{\frac{1}{s} + \frac{1}{\sigma}}{\frac{1}{s^2} + \frac{1}{\sigma^2}} = \frac{s\sigma(s + \sigma)}{s^2 + \sigma^2}.$$

(2) Vols. proportional to $s . \sigma$ are mixed.

$$\text{Sp. gr. of mixture} = \frac{s^2 + \sigma^2}{s + \sigma}.$$

$$\therefore\ \text{Ratio of sp. grs.} = \frac{s\sigma(s + \sigma)^2}{(s^2 + \sigma^2)^2}.$$

CHAPTER III.

EXAMINATION.

1. THE pressure is increased by the weight of the liquid contained in a cylinder whose base is equal to that of the vessel and whose height is equal to the increase of depth of the liquid.

2. (1) The pressure on a unit of area (a square inch) = the weight of $\frac{100}{144}$ cubic feet of water $= 43\frac{29}{72}$ lbs. wt.

(2) Pressure is now increased by the atmospheric pressure, i.e. by about $14\frac{1}{2}$ lbs. wt. per square inch, and is therefore about 58 lbs. wt.

4. Neglecting atmospheric pressure, the pressure on a square inch = the weight of $\frac{170}{144}$ cubic feet of water $= 73\frac{113}{144}$ lbs. wt.

6. The depth of the centre of gravity of the triangle is $\dfrac{1}{2\sqrt{3}}$ feet.

Its area is $\sqrt{3}/4$ square feet.

$\therefore$ the pressure on it = the weight of $\frac{1}{8}$ cubic foot of water = weight of 125 ozs.

8. If a cylinder with vertical generating lines be described on the same base as the cone, the pressure on the curved surface of the cone is the weight of the liquid which would fill the space between the cone and cylinder, i.e. twice the weight of the liquid in the cone.

9. If we suppose the density of the area to vary as the depth below the surface, its centre of gravity would be the centre of pressure. Now this centre of gravity is evidently at a greater depth than that of a uniform area with the same boundary, unless the area be horizontal, when they coincide.

10. Let $2a$ be the height of the rectangle, $2b$ its breadth and let h be the depth of the upper edge.

The pressure when this edge is in the surface is $4a^2bw$ at a point at depth $4a/3$, taking w as the intrinsic weight of the liquid.

This pressure is increased by the pressure $4abhw$ uniformly distributed, the resultant of which may therefore be considered as acting at the c. g.

If x be the depth below the upper edge of the consequent resultant pressure

$$4abw(a+h)x=4a^2bw\cdot\frac{4a}{3}+4abhw\cdot a;$$

$$\therefore\ x=\frac{a}{a+h}\left[\frac{4a}{3}+h\right]$$
$$=a+a^2/3(a+h).$$

The centre of pressure is therefore at a depth $a^2/3(a+h)$ below the centre of gravity.

11. Let h be the length of the vertical side of the rectangle, x the depth of the required line, b the breadth of the rectangle.

The whole pressure on the rectangle $=w\cdot\frac{h}{2}\cdot hb$.

The pressure on the upper portion $=w\cdot\frac{x}{2}\cdot xb$.

$$\therefore\ x^2=\tfrac{1}{2}h^2 \text{ or } x=h/\sqrt{2}.$$

12. If x_r be the depth of the rth line of division, we have as in (11),

$$x_r^2=\frac{r}{n}h^2;$$

$$\therefore\ x_r=\frac{r}{\sqrt{n}}h.$$

13. h being the height, $2a$ the base of the triangle, x the depth of the dividing line, whose length $\therefore=2\cdot\frac{xa}{h}$.

Whole pressure on triangle $=w\cdot\tfrac{2}{3}h\cdot ha$.

Pressure on upper portion $=w\cdot\tfrac{2}{3}x\cdot x\frac{xa}{h}$;

$$\therefore\ x^3/h=\tfrac{1}{2}h^2;$$
$$\therefore\ x=h/\sqrt[3]{2}.$$

Examples.

1. Let A be the sectional area of the cylinders, W the weight of the piston, h, k the heights of the water in the open and closed cylinders. w being the intrinsic weight of water the position of equilibrium is given by

$$w \,.\, hA = wkA + W,$$

or

$$h - k = W/wA.$$

The volume of water $(h+k)\,A$ in the two cylinders being known, h and k are at once determined.

2. The weight must be equal to the weight of $8 \times (2\frac{1}{2})^2$ cubic feet of water, i.e. 3125 lbs. wt.

3. Let h be the height, b the breadth of the rectangle, and let the line cut the lower side at a distance $b - x$ from the opposite corner.

Pressure on whole rectangle $= w \,.\, \dfrac{h}{2} \,.\, hb.$

Pressure on triangular part $= w \,.\, \dfrac{2h}{3} \,.\, \dfrac{hx}{2}.$

$$\therefore\ \frac{2x}{3} = \tfrac{1}{2}b \text{ or } x = \tfrac{3}{4}b.$$

4. Let h be the whole length of tube occupied by the two liquids, x the height of their common surface above B.

The height of the surface of the lighter liquid above the common surface $= h/2\sqrt{2}$.

That of the surface of the heavier liquid is

$$\left(\frac{h}{2} - 2x\sqrt{2}\right) \Big/ \sqrt{2};$$

$$\therefore\ 2\left(\frac{h}{2} - 2x\sqrt{2}\right) = \frac{h}{2} \text{ or } x = h/8\sqrt{2}.$$

5. Area of curved surface $= \pi$ square feet.

Pressure on it due to weight of water $=$ weight of $\dfrac{\pi}{2}$ cubic feet of water.

Pressure on it due to weight of piston $= \pi \times 4 \div \dfrac{\pi}{4} = 16$ lbs. wt.

$\therefore$ whole pressure $= 16 + \frac{125}{4}\pi$ lbs. wt.

6. The cylinder being full of water, the effect is simply that which would be produced by increasing the weight of the piston by 1 lb. wt., i.e. the whole pressure is increased by 4 lbs. wt.

The pressure at a depth h is $\dfrac{20}{\pi} + \dfrac{125}{2} h$ lbs. wt. per square foot.

7. (1) When the vessel is full, the water will overflow as the lead is immersed, the pressure on the base being unchanged.

(2) If the vessel is not full, the pressure on the base is increased by the weight of a quantity of water equal in volume to the piece of lead.

8. Let h be the height, r the radius of the cylinder.

Pressure on curved surface $= wrh \,.\, 2\pi r$.

Pressure on each end $= wr \,.\, \pi r^2$.

Weight of fluid $= wh \,.\, \pi r^2$.

$$\therefore\ 2hr + 2r^2 = 3hr,$$

or

$$2r = h.$$

9. Let the vertical through C meet AB in D, and let θ be the angle made by CA, CB with the surface.

The depth of A is $b \sin\theta$, that of B is $a \sin\theta$;

$$\therefore \text{ the depth of } D = \frac{2ab \sin\theta}{a+b}.$$

For CD bisects ACB and therefore $AD : DB = b : a$.

The depth of the c.g. of ACD

$$= \tfrac{2}{3}\left(b \sin\theta + \frac{2ab \sin\theta}{a+b}\right)\Big/ 2 = \frac{\sin\theta}{3} \,.\, \frac{b(b+3a)}{a+b}.$$

The depth of the c.g. of $BCD = \dfrac{\sin\theta}{3} \,.\, \dfrac{a(a+3b)}{a+b}$.

The areas of ACD, BCD are as $AD : BD = b : a$;

$\therefore$ the pressures on them are as $b^2(b+3a) : a^2(a+3b)$.

10. Let w, $3w$ be the intrinsic weights of the two fluids, h the depth of each, r the radius of the cylinder.

The whole pressure on the surface in contact with the lighter fluid is

$$w \,.\, 2\pi rh \,.\, \frac{h}{2} = w\pi rh^2.$$

The whole pressure on the curved surface in contact with the heavier fluid is

$$wh \,.\, 2\pi rh + 3w \,.\, 2\pi rh \,.\, \frac{h}{2} = 5w\pi rh^2.$$

$\therefore$ the whole pressure on the curved surface $= 6w\pi rh^2$.

If the fluids be mixed, the intrinsic weight of the mixture will be $2w$.

The whole pressure on the curved surface will be

$$2w \,.\, 2\pi r \,.\, 2h \,.\, h = 8w\pi rh^2.$$

and $$8w\pi rh^2 : 6w\pi rh^2 = 4 : 3.$$

11. Let ABC be the triangle, AB being in the surface. If O be any point in the line CD, joining C to the middle point of AB,

The pressures on the triangles CAD, CBD are equal and the pressures on OAD, OBD are equal.

$\therefore$ the pressures on COA, COB are equal.

If $OD = x$ and $CD = h$,

Pressure on OAB : pressure on $CAB = x^2 : h^2$.

$\therefore$ if O be the point required,

$$x^2 = \tfrac{1}{3}h^2 \quad \text{or} \quad x = h/\sqrt{3}.$$

12. Let $$AD = x.$$

Then pressure on ABD : pressure on $ABC = x^2 : b^2$;

$$\therefore\ x^2 = \tfrac{1}{2}b^2.$$

And $$AD : DC = x : b - x = 1 : \sqrt{2} - 1.$$

13. Let x inches be the side of the square, $2w$, $3w$ the intrinsic weights of the two fluids.

The pressure on the part in the lighter fluid

$$= 2w \,.\, 2 \,.\, 4x = 16wx.$$

That on the lower part $= \left(2w \,.\, 4 + 3w \,.\, \dfrac{x-4}{2}\right) x\,(x-4)$.

These being equal, we obtain

$$x = \tfrac{4}{3}\,(1 \pm \sqrt{10}).$$

The upper sign must be taken, since x is essentially positive.

14. Let p be the perimeter of the cylinder, h the depth of each fluid.

The fluid pressures on the three portions are as

$$\frac{h}{2} \cdot ph : \left(h + 2 \,.\, \frac{h}{2}\right) ph : \left(h + 2h + 3\,\frac{h}{2}\right) ph;$$

$\therefore$ these pressures are as $1 : 4 : 9$.

15. Let, α, $\pi - \alpha$, be the angles subtended at the centre of the tube by the two portions of fluid, ρ, ρ' their densities, a the radius of the circular tube, θ the inclination to the vertical of the bounding diameter.

The two expressions for the pressure at the common surface are in the ratio of

$$\rho\,[a\cos\theta - a\cos(\theta+\alpha)],$$

and

$$\rho'\,[-a\cos(\theta+\alpha) - a\cos\theta].$$

Equating these we find

$$\tan\theta = \cot\alpha + \frac{1}{\sin\alpha}\frac{\rho'+\rho}{\rho'-\rho}.$$

16. Let h be the length of the cylinder, A its sectional area, θ, $\frac{\pi}{2}-\theta$, the inclinations of the axis to the vertical, w the intrinsic weight of the fluid.

$$P = wA\,.\,h\cos\theta,$$

$$P' = wA\,.\,h\sin\theta;$$

$\therefore\ [P^2+P'^2]^{\frac{1}{2}} = whA =$ weight of fluid displaced.

17. Let w be the intrinsic weight of the fluid, r the radius of the cylinder $\therefore$ $2r$ the depth of fluid.

Pressure on curved surface $= w\,.\,2\pi r\,.\,2r\,.\,r = 4w\,.\,\pi r^3$.

Let h be the distance of the centre of the sphere from the base.

$$\frac{2}{3}\pi r^3 + \pi r^2\,.\,2r = \pi r^2\,.\,h \qquad \therefore\ h = \frac{8}{3}r.$$

One eighth of the sphere's weight is supported by the displacement of fluid, the rest by pressure on the surface of the fluid, which is therefore $\frac{7}{8}\,4w\,.\,\frac{4}{3}\pi r^3 = \frac{14}{3}\,w\pi r^3$ or $\frac{14}{3}\,wr$ per unit of area.

$\therefore$ Pressure on curved surface is

$$\left[\frac{14}{3}\,wr + w\,.\,\frac{4}{3}\,r\right]2\pi r\,.\,\frac{8r}{3} = 32w\pi r^3.$$

The increase $= 28w\pi r^3$ or seven times the original pressure or 21 times the weight of the fluid.

18. Let ρ, ρ', ρ'' be the three densities,

x, y the depths of the common surfaces.

Then

$$\rho x = \rho'' y - \rho'(y-x),$$

or

$$(\rho-\rho')\,x = (\rho''-\rho')\,y;$$

$\therefore$ $\rho-\rho'$ and $\rho''-\rho'$ are of the same sign.

And if $y>x$, $\rho''-\rho'<\rho-\rho'$;

$\therefore$ the order of magnitude is $\rho\,.\,\rho''$, ρ'.

And these being in A.P.,

$$\rho-\rho' = 2\,(\rho''-\rho');$$

$$\therefore\ y = 2x.$$

19. Referring to Examination (9), the area whose C.G. is the centre of pressure becomes more nearly of uniform density as the plane area is lowered. Hence its c.g. approaches and ultimately coincides with that of the area of uniform density.

20. The whole pressure on the area is originally $w \cdot \pi r^3$. This is increased by a pressure wr per unit area uniformly distributed, the resultant being therefore $w\pi r^3$ at the centre.

The two equal forces at the centre of pressure and centre of figure have a resultant whose line of action bisects the line joining these two points.

21. Let x be the distance of the centre of pressure from the centre of the square.

The old pressure being $\frac{1}{2} wa^3$ at a depth $\frac{a}{6}$ below the centre, and the increase of pressure being wa^2b we have

$$\left[\frac{1}{2} wa^3 + wa^2b\right] x = \frac{1}{2} wa^3 \cdot \frac{a}{6}$$

$$w = a^2/(6a + 12b).$$

22. Produce AB, DC to meet in E.

Let $DE = h\alpha$ and $\therefore$ $CE = h\beta$.

Pressure on $ECB = w \cdot \frac{\beta}{3} \cdot \frac{\beta^2 h}{2}$, and depth of centre of pressure $= \frac{\beta}{2}$.

Pressure on $EDA = w \cdot \frac{\alpha}{3} \cdot \frac{\alpha^2 h}{2}$, and depth of centre of pressure $= \frac{\alpha}{2}$.

$\therefore$ depth of centre of pressure of $ABCD$

$$= \frac{\alpha^3 \cdot \frac{\alpha}{2} - \beta^3 \cdot \frac{\beta}{2}}{\alpha^3 - \beta^3} = \frac{1}{2}\left(\frac{\alpha^3 + \alpha^2\beta + \alpha\beta^2 + \beta^3}{\alpha^2 + \alpha\beta + \beta^2}\right).$$

23. Let x be the height of the lower portion, h that of the triangle, w the intrinsic weight of water.

The areas are as $x^2 : h^2$.

The depth of the c.g. of the upper portion is

$$\frac{h^2 \cdot \frac{h}{3} - x^2\left[h - x + \frac{x}{3}\right]}{h^2 - x^2} = \frac{1}{3} \cdot \frac{h^3 - 3hx^2 + 2x^3}{h^2 - x^2}.$$

The pressure on it is $\frac{1}{3} w (h^3 - 3hx^2 + 2x^3)$.

That on the lower portion is

$$w\left[(h - x) + 14\frac{x}{3}\right] x^2 = \frac{1}{3} wx^2 (3h + 11x).$$

Equating these we obtain

$$(h-3x)(h^2+3hx+3x^2)=0.$$

The real root $x=\frac{h}{3}$ gives the solution of the present problem.

$\therefore$ the areas of the two portions are as

$$h^2-x^2 : x^2=8 : 1.$$

24. Let $2h$ be the height of the floodgate, b its width, P, p the pressures at the upper and lower corners.

$$2(P+p)=wh.2hb-w.\frac{h}{2}.hb=\tfrac{3}{2}wh^2b,$$

$$2p.2h=wh.2hb.\frac{4h}{3}-w\frac{h}{2}.hb.\frac{2h}{3}=\tfrac{7}{3}wh^3b;$$

$$\therefore\ p=\tfrac{7}{12}wh^2b.$$
$$P=\tfrac{1}{6}wh^2b.$$

25. Let $\rho.\rho'$ be the densities, h the depth of each liquid.

Then $$3\rho.\frac{h}{2}=\rho h+\rho'\frac{h}{2};$$

$$\therefore\ \rho=\rho'.$$

26. The area of the curved surface $=2\pi r^2$, r being the radius. Its c.g. is at a depth $\frac{r}{2}$.

$\therefore$ whole pressure on it $=w\pi r^3$.

Area of base $=\pi r^2$ and pressure per unit area on it is wr, $\therefore$ whole pressure $=w\pi r^3=$ that on the curved surface.

27. Let a be the length of a side of the square, and let the two lower sides be produced to meet the surface of the liquid.

The area of the triangle so formed is $2a^2$ and its centre of pressure is at a depth $a/\sqrt{2}$, the whole pressure being

$$w.2a^2.\frac{\sqrt{2}a}{3}=\frac{2\sqrt{2}}{3}wa^3.$$

Each of the two triangles added is of area $a^2/2$ and has the centre of pressure at a depth $a/2\sqrt{2}$, the whole pressure on each being $\frac{1}{6\sqrt{2}}wa^3$.

If x be the depth of the centre of pressure of the square,

$$\frac{1}{\sqrt{2}}wa^3.x+\frac{1}{12}wa^4=\frac{2}{3}wa^4.$$

$$x=\frac{7\sqrt{2}}{12}a.$$

$$=\frac{7}{12}\text{ diagonal of square.}$$

28. All the pressures on the curved surface of the cone are inclined at the same angle a to the horizontal, a being the semi-vertical angle of the cone.

If therefore P be the whole pressure, its vertical component is $P\sin a$, and this is the resultant pressure.

It is therefore equal to the weight of the liquid which would fill the space between the cone and a cylinder on the same base with vertical generating lines.

This weight is unaltered by mixing up the fluid and $\therefore P$ is unaltered by this process.

29. If AB be the side in the surface CD the parallel side, by drawing one diagonal as AC we separate the trapezium into two triangles the centres of pressure on which are at depths

$$\frac{h}{2}\text{ (for } ABC\text{) and }\frac{3b}{4}\text{ (for } ACD\text{).}$$

The whole pressures on them are $\frac{1}{6}w\,.\,ah^2$ and $\frac{1}{3}wbh^2$.

If x be the depth of the centre of pressure of the trapezium, we have

$$x\left[\frac{1}{6}wah^2+\frac{1}{3}wbh^2\right]=\frac{1}{6}wah^2\,.\,\frac{h}{2}+\frac{1}{3}wbh^2\,.\,\frac{3h}{4}.$$

$$2x(a+2b)=(a+3b)h$$

or

$$x=\frac{a+3b}{a+2b}\cdot\frac{h}{2}.$$

30. Let x be the depth of the plane of division h the height of the cone, $2a$ the vertical angle.

Whole pressure on curved surface of cone

$$=w\,.\,\pi h^2\tan a\sec a\,.\,\frac{2h}{3}=\frac{2}{3}w\pi h^3\tan a\sec a.$$

On the upper part it is $\frac{2}{3}w\pi x^3\tan a\sec a$.

$$\therefore\ x^3=\frac{1}{2}h^3\ \text{ or }\ x=h/\sqrt[3]{2}.$$

In the second case, z being the distance from the vertex of the plane of division,

Whole pressure on curved surface of cone

$$=w\,.\,\pi h^2\tan a\sec a\,.\,\frac{h}{3}=\frac{1}{3}w\pi h^3\tan a\sec a.$$

Pressure on lower part

$$=w\,.\,\pi z^2\tan a\sec a\,.\left(h-\frac{2z}{3}\right).$$

$$\therefore\ z^2\left(h-\frac{2}{3}z\right)=\frac{1}{6}h^3.$$

$$2z^3-3z^2h-\frac{1}{2}h^3=0.$$

$$(2z-h)\left(z^2-zh+\frac{h^2}{2}\right)=0.$$

$$\therefore\ z=\frac{h}{2}\text{ the other values being imaginary.}$$

31. Let a be the radius of the tube, $\omega_1, \omega_2, \omega_3$, the intrinsic weights of the fluids, $p_\alpha, p_\beta, p_\gamma$ the pressures at the three surfaces of separation.

These surfaces are at heights above the centre $a\cos a$, $a\cos\beta$, $a\cos\gamma$ respectively,

$$\therefore\quad p_\beta-p_\alpha=\omega_3 a(\cos a-\cos\beta)$$

$$p_\gamma-p_\beta=\omega_1 a(\cos\beta-\cos\gamma)$$

$$p_\alpha-p_\gamma=\omega_2 a(\cos\gamma-\cos a).$$

Adding, and remembering that $\omega_1, \omega_2, \omega_3$ are proportional to ρ_1, ρ_2, ρ_3 we obtain

$$\rho_1(\cos\beta-\cos\gamma)+\rho_2(\cos\gamma-\cos a)+\rho_3(\cos a-\cos\beta)=0.$$

If there are equal quantities of each fluid, and a refers to the highest point of junction

$$\beta-a=120^0,\quad \gamma+a=120^0.$$

And the weights on each side of the vertical diameter are equal,

$$\therefore\quad \rho_2 a+\rho_3\,.\,120^0+\rho_1(60^0-a)$$

$$=\rho_2(120^0-a)+\rho_1(60^0+a)$$

or

$$(\rho_2-\rho_3)\,60^0=(\rho_2-\rho_1)\,a.$$

And the former equation gives

$$\rho_1\sin a+\rho_2\sin(60^0-a)-\rho_3\sin(60^0+a)=0.$$

If $a=30^0$ both equations are satisfied and $\rho_1+\rho_2=2\rho_3$ so that ρ_1, ρ_3, ρ_2, form an A. P.

32. Let a, b, c be the breadths of the sides of the prism, δ, ϵ, ζ the ~~lengths~~ of the edges.

P is proportional to $a\dfrac{\epsilon+\zeta}{2}$, i.e. to $\sin\alpha\dfrac{\epsilon+\zeta}{2}$,

Q to $\sin\beta\,.\,\dfrac{\zeta+\delta}{2}$, R to $\sin\gamma\dfrac{\delta+\epsilon}{2}$.

$\therefore$ $P\operatorname{cosec}\alpha+Q\operatorname{cosec}\beta+R\operatorname{cosec}\gamma$ is proportional to $\delta+\epsilon+\zeta$, which being three times the depth of the c.g. of the prism remains unchanged.

33. Let a be the length of the edge of the cube,

h the depth of the fluid in it,

W the weight of a side, w the intrinsic weight of the fluid.

The width of the surface of the fluid $=a-h$;

$\therefore$ the volume of the fluid $=ah\,.\,\dfrac{a+\overline{a-h}}{2}=\frac{1}{2}ah\,(2a-h)$.

Since this $=\frac{1}{4}a^3$, $h=a\left(1-\dfrac{1}{\sqrt{2}}\right)$.

The pressure on the loose face is $w\,.\,\dfrac{h}{2}\,.\,ah\sqrt{2}$.

Its moment about the hinge $=w\dfrac{ah^2}{\sqrt{2}}\,.\,\dfrac{h\sqrt{2}}{3}$.

The moment of the weight of the face $=W\,.\,\dfrac{a}{2\sqrt{2}}$.

Equating these and inserting the value of h,

$$\frac{W}{\frac{1}{4}wa^3}=\frac{4}{3}(\sqrt{2}-1)^3=\frac{4}{3}(5\sqrt{2}-7).$$

34. Let the box be tilted through an angle θ.

(i) About the edge on the same side as the hinge.

Pressure on lid $=w.\dfrac{a\sin\theta}{2}\,.\,a^2$.

Its moment about the hinge $=w\,.\,\dfrac{a^3\sin\theta}{2}\,.\,\dfrac{2a}{3}$.

Moment of weight of lid about hinge $=W.\dfrac{a\cos\theta}{2}$;

$\therefore$ when $\tan\theta=\frac{3}{2}\dfrac{W}{wa^3}$ the water begins to escape.

(ii) About the edge diagonally opposite to the hinge.

Moment of fluid pressure about hinge $=w.\dfrac{a^3\sin\theta}{2}\,.\,\dfrac{a}{3}$;

$\therefore$ when $\tan\theta=3\,.\,\dfrac{W}{wa^3}$ the water begins to escape.

(iii) About either of the other edges.

Moment of fluid pressure about hinge $= w \, . \, \dfrac{a^3 \sin\theta}{2} \, . \, \dfrac{a}{2}$.

Moment of weight about hinge $= W \cos\theta \, . \, \dfrac{a}{2}$.

$\therefore$ When $\tan\theta = 2 \dfrac{W}{wa^3}$ the water begins to escape.

These values of $\tan\theta$ are as $3 : 6 : 4$.

35. The volume of the wine $= \frac{2}{3} \times \frac{12}{11} = \frac{8}{11}$ that of the water.

The density of the mixture $= \dfrac{8 \, . \, 11 + 11 \, . \, 12}{8 + 11} = \dfrac{220}{19}$.

Taking the density of water as 12,

The depth of pure wine $= \frac{4}{19}$ of the cylinder,

That of the mixture $= \frac{1}{2}$ of the cylinder, and of water, $\frac{11}{38}$ of the cylinder,

The pressure on the curved surface in contact with the water is proportional to

$$\left(11 \times \tfrac{4}{19} + \tfrac{220}{19} \, . \, \tfrac{1}{2} + 12 \, . \, \tfrac{11}{76}\right) \tfrac{11}{38}.$$

That on the rest of the surface is proportional to

$$11 \, . \, \tfrac{2}{19} \, . \, \tfrac{4}{19} + \left(11 \, . \, \tfrac{4}{19} + \tfrac{220}{19} \, . \, \tfrac{1}{4}\right) \tfrac{1}{2}.$$

And these two quantities are equal.

36. Let A be the area of the fluid surface, h the depth of the vertex below that surface in the first case.

The area of surface in contact with fluid is $\dfrac{A \cos\theta}{\sin\alpha}$.

$\therefore$ Fluid pressure on it $= w \dfrac{h}{3} \, . \, \dfrac{A \cos\theta}{\sin\alpha}$.

Let A' be the area of the fluid surface, k the depth of the fluid when the cone is vertical.

The fluid pressure $= w \, . \, \dfrac{k}{3} \, . \, \dfrac{A'}{\sin\alpha}$.

But $A' \, . \, k = A \, . \, h$ since the volume of fluid is the same.

$\therefore$ The pressure is changed in the ratio $\cos\theta : 1$.

37. The resultant pressure on the curved surface is due to the weight of the fluid and the upward pressure of the base, which are equal and opposite parallel forces of magnitude W, and the distance between their lines of action is $\frac{1}{2} h \tan\alpha$.

CHAPTER IV.

EXAMINATION.

5. THE c. g. of the two weights must be at the middle of the plank.

7. One-third of the cylinder must be immersed;

$$\therefore \text{ its length is 12 feet.}$$

8. The specific gravities of the fluids are respectively $\frac{4}{3}$ and $\frac{5}{4}$ that of the solid, and are $\therefore$ as 16 : 15.

9. In each case the force is equal to the weight of the water displaced by six inches of the cylinder, i.e. by one-third of its own weight.

10. One-half the perimeter of the triangle must be immersed, i.e. 3/4 of each of the two lower sides.

12. One-half the sphere is immersed, equilibrium being established with the nail at the highest point (unstable) or at the lowest point (stable).

13. The density of ice being about nine-tenths of the density of water, it is clear that the centre of the sphere is below the surface of the water.

Assuming that the spherical form is maintained, the sphere will melt down to a particle in the surface. The centre of the sphere there fore rises.

14. If x cubic feet be the required volume:

Volume of water displaced $=4+x$ cubic feet.

$$\therefore 4+x=4\,.\,\tfrac{1}{2}+x\,.\,\tfrac{7}{2}.$$

$$\therefore x=\tfrac{4}{5}.$$

15. The pressure is that of the wood and superincumbent volume of water, which is of equal volume with the wood.

16. The water will flow from that vessel in which the pressure at the base is the greater, and in order that the equilibrium may not be disturbed, the heights of the liquids must be inversely as their intrinsic weights.

17. Let V, V', be the volumes, σ, σ' their specific gravities, s the sp. gr. of water.

The weights of the two bodies in water being equal

$$V(\sigma - s) = V'(\sigma' - s),$$

$$\text{or } \frac{V}{V'} = \frac{\sigma' - s}{\sigma - s}.$$

Examples.

1. The volume immersed is equal to that not immersed, and the c. g. of the part immersed coincides with that of the fluid displaced, so that the c. g.'s of the two halves are in a vertical line. Hence, if inverted, the solid will float in equilibrium.

2. Let the volume of the granite be x cubic yards.

$$\cdot 918(1 - x) + 2\cdot 65x = \tfrac{23}{25} = \cdot 92.$$

$$\therefore x = \tfrac{1}{866}.$$

3. Half the area of the triangle must be immersed.

$\therefore$ The altitude is divided by the surface of the liquid in the ratio

$$1 : \sqrt{2} - 1.$$

4. Half the volume of the cone being immersed in each case, the portions of the axis immersed are in the ratio

$$1 : \sqrt[3]{2} - 1.$$

5. Let w be the weight of the body.

Then $$w - w_1 : w - w_2 : w - w_3 :: s_1 : s_2 : s_3.$$

$$\therefore w_1(s_2 - s_3) + w_2(s_3 - s_1) + w_3(s_1 - s_2) = 0.$$

6. $\frac{1}{4}$ of AB is above the surface and $\therefore$ $\frac{1}{8}$ of the triangle.

If its density be ρ, and that of the liquid unity, taking moments about A of the weight of the lamina and of the displaced fluid

$$\rho = 1 - \tfrac{1}{8} \cdot \tfrac{1}{2} = \tfrac{15}{16}.$$

7. Let h be the depth of the upper liquid, x the length of the cylinder in the lower liquid, $\therefore$ $2h - x$ that in the upper.

$$\frac{7\rho}{4} \cdot 2h = (2h - x)\rho + x \cdot 2\rho,$$

giving $x = \frac{3}{2}h$, i.e. one-quarter of the cylinder is in the upper liquid.

8. The c. g. of the weighted rod is one-third of its length from the weighted end.

$\therefore$ The c. g. of the displaced liquid is at this point.

$\therefore$ Two-thirds of the length of the rod is immersed and the density of the liquid $= \frac{3}{2}$ density of weighted rod $= \frac{9}{4}$ that of the wood.

9. Let U be the volume not immersed, V the volume immersed, s the sp. gr.

Then Us is given and $(U+V)s = V \times 1$;

$$\therefore \quad U+V \propto \frac{1}{s} + \frac{1}{1-s} \propto \frac{1}{s-s^2},$$

which is least when $s = \frac{1}{2}$.

10. Weight of water to be displaced is that of

$$\pi \cdot \frac{9}{4} \cdot \frac{9}{2} \cdot \frac{1000}{1728} = \frac{375\pi}{64} \text{ ozs.}$$

$\therefore$ The required weight $= \left(\frac{375\pi}{64} - 8\right)$ ozs. wt.

11. Evidently one-half that in the previous example.

12. The c. g. of the weighted rod, and $\therefore$ of the displaced fluid, is one-quarter the length of the rod from the weighted end.

$\therefore$ One-half the rod is immersed.

$\therefore$ The weight of fluid displaced by half the rod is double the weight of the rod.

$\therefore$ The density of the fluid is four times that of the rod.

13. The tension of the string $=$ weight of rod $-$ weight of water displaced, and is therefore not changed when the inclination is changed.

14. Let r, r' be the internal and external radii, σ, ρ the densities of the shell and of water.

$$\tfrac{4}{3}\pi (r'^3 - r^3)\sigma = \tfrac{2}{3}\pi r'^3 \rho.$$
$$\therefore \quad \sigma : \rho = r'^3 : 2(r'^3 - r^3).$$

15. Let w, w' be the weights of the cone and of the displaced fluid. $w - w'$ is the magnitude of the required force.

If x be the distance of its line of action from the c. g. of the solid cone,

$$(w-w')x = w\left(\frac{3}{4}h - \frac{2}{3}h\right) = w\frac{h}{12}.$$

$$\therefore \quad x = \frac{w}{w-w'}\frac{h}{12}.$$

16. $\frac{19}{27}$ of the volume of the cone being immersed, $\frac{8}{27}$ of it is above the surface, i.e. $\frac{2}{3}$ of the axis is above the surface.

17. Let σ, ρ, be the intrinsic weights of the lamina and fluid, $2A$ the area of the lamina, w the weight attached.

BD being the perpendicular from B on AC, let $AC=d$, $AD=x$.

$$w+2A\sigma=A\rho.$$

The distance of A from the vertical through the c. g. of the half immersed is

$$\frac{2}{3}\left[\frac{x+d}{2}\right].$$

$$\therefore\ A\rho\,.\,\frac{x+d}{3}=wx+2A\sigma\,.\,\frac{d}{2}.$$

$$\therefore\ \rho=3\sigma.$$

18. Let σ, ρ, be the intrinsic weights of paraboloid and liquid, h the height of the paraboloid, x that of the portion immersed. Area of base of paraboloid varies as the height.

$$\therefore\ \rho x^2=\sigma h^2$$

or
$$x=h\sqrt{\frac{\sigma}{\rho}}.$$

19. Let W be the weight of ship and cargo, in tons wt.

V the volume immersed, in cubic feet.

A the area of the water line section, in square feet.

$$(1.025)\ V.\frac{62\cdot5}{2240}=W=\left(V+\frac{A}{6}\right)\frac{62\cdot5}{2240},$$

$$A\,.\,\frac{3}{24}\,.\,\frac{62\cdot5}{2240}=40.$$

Eliminating A and V we obtain $W=2186\frac{2}{3}$.

20. Let $2k$ be the height of the required cylinder.

Half of it being immersed,

$$\pi\left(r^2-r'^2\right)k=\tfrac{2}{3}\pi r^2\,.\,h-\pi r^2\,(h-k).$$

$$\therefore\ 2k=\frac{r^2}{r'^2}\,.\,\frac{h}{2}.$$

21. The pressure on the lower half of the curved surface is vertical, and bears to the weight of the water in the cylinder the ratio,

$$\frac{\pi r^2}{2}+2r^2:\pi r^2=\frac{1}{2}+\frac{2}{\pi}:1,$$

since it is in equilibrium with the weight of the water in the lower half and the pressures on the horizontal plane separating the two halves. The latter pressure is that due to a depth r of water all over the surface of separation. When the cylinder is vertical, the pressure on one-half the curved surface is horizontal and balances the pressure on the diametral plane. It acts $\therefore$ at a point in the axis $\frac{2}{3}$ of its length from the top, and is to the weight of the water contained in the cylinder as twice the height to the perimeter.

22. The greatest height is when the c. g. of the whole solid coincides with the centre of the hemisphere.

h being the height of the cylinder, r its radius

$$\pi r^2 h \cdot \frac{h}{2} = \frac{2}{3}\pi r^3 \cdot \frac{3r}{8}.$$

$$\therefore\ h = r/\sqrt{2}.$$

23. Let V be the whole volume of the body.

σ, ρ, the intrinsic weights of the body and fluid.

$$V\sigma = P_1\rho_1 + (V - P_1)\rho = P_2\rho_2 + (V - P_2)\rho = P_3\rho_3 + (V - P_3)\rho.$$

$$\therefore\ \frac{\rho_2 - \rho_3}{P_1} + \frac{\rho_3 - \rho_1}{P_2} + \frac{\rho_1 - \rho_2}{P_3} = 0.$$

24. The volume of water displaced by box and float is 5·34 cub. ins. The weight of the float is equal to that of 2·17 cub. ins. of water, therefore that of the box is equal to that of 3·17 cub. ins. of water.

The volume of metal is

$$1 - (\tfrac{8}{9})^3 = \tfrac{217}{729} \text{ cub. in.}$$

$\therefore$ Its sp. gr. is $3{\cdot}17 \div \frac{217}{729} = 10{\cdot}557\ldots$

25. Let A square feet be the area of the section in the neighbourhood of the water line, V the volume immersed at first, and x the rise in feet in passing from fresh to sea water.

Then
$$A \cdot \frac{1}{12} \cdot \frac{62{\cdot}5}{2240} = 30,$$

$$\frac{62{\cdot}5}{2240} V = \frac{64}{2240}(V - xA) = W,$$

and
$$600 = (2 - x)\frac{64A}{2240}.$$

It will be found from these equations that

$$x = \frac{143}{384}, \text{ and } W = 5720.$$

26. Volume of frustum $= \frac{7}{8}$ that of complete cone.

Volume of part immersed $= \frac{27}{64} - \frac{1}{8} = \frac{19}{64}$ that of cone.

$\therefore$ Density of cone : that of fluid $= 19 : 56$.

27. Expressing the same condition as in (22),

$$\frac{1}{3}\pi r^2 . h . \frac{h}{4} = \frac{2}{3}\pi r^3 . \frac{3r}{8}.$$

$$\therefore h = \sqrt{3} . r.$$

28. σ, ρ, being intrinsic weights of rods and fluid.

a the length of each rod, α the sectional area.

Moment of fluid pressures about hinge

$$= a\alpha . \rho . \frac{a}{2} . \frac{1}{\sqrt{5}} + a\alpha\rho\left(\frac{a}{\sqrt{5}} + \frac{a}{2} . \frac{2}{\sqrt{5}}\right) + \frac{a\alpha\rho}{2}\left(\frac{a}{4} . \frac{1}{\sqrt{5}} + \frac{a\sqrt{5}}{2}\right)$$

$$= a^2\alpha\rho \frac{31}{8\sqrt{5}}.$$

Moment of weight of rods about hinge

$$= a^2\alpha\sigma\left(\frac{1}{2\sqrt{5}} + \frac{2}{\sqrt{5}}\right) + a\alpha\sigma . \frac{a\sqrt{5}}{2} = a^2\alpha\sigma\sqrt{5}.$$

$$\therefore \sigma : \rho = 31 : 40.$$

29. Let the surface of the fluid meet AC in D.

Since the pressures of the fluid and the weight of the triangle act vertically through the c. g.'s of BCB, BAC respectively, these c. g.'s are in a vertical line. $\therefore AC$ is vertical, and $\therefore$ perpendicular to BD.

And density of fluid : density of triangle

$$= AC : CD = AC : CB \cos C = \sin B : \sin A \cos C.$$

30. Since all the pressures on the curved surface make the same angle with the vertical, the whole pressure and the resultant pressure are proportional.

Now the resultant pressure is the weight of the cone, $\therefore$ the whole pressure is constant.

31. Let σ, ρ, ρ', be the intrinsic weights of the solid and of the two fluids, h the height of the cylinder, x the length of it in the lower fluid.

$$(h - x)\rho + x\rho' = h\sigma,$$

$$x = h . \frac{\sigma - \rho}{\rho' - \rho}.$$

If ρ be increased, x is diminished, i.e. the cylinder rises. If x becomes $x+\delta$, the upward pressure on the cylinder is proportional to

$$(h-x-\delta)\rho+(x+\delta)\rho'=h\sigma+\delta(\rho'-\rho),$$

i.e. such as to urge the cylinder back towards equilibrium. $\therefore$ the equilibrium is stable.

32. Let the surface of the fluid intersect the rods in D, E.

Let P be the stress at B in a vertical direction, σ, ρ, the intrinsic weights of the rods and fluid.

Then $$P+\rho\,.\,CE=\sigma\,.\,BC.$$

And, taking moments about B for the rod BC,

$$\sigma\,.\,BC\,.\,\frac{BC}{2}=\rho\,.\,CE\left(BE+\frac{CE}{2}\right)$$

$$\therefore\ \rho:\sigma=BC:CE\left(1+\frac{BE}{BC}\right)=BC^2:BC^2-BE^2,$$

$$P=\rho\,\frac{BE\,.\,CE}{BC}.$$

Taking moments about A for the rod AB,

$$P\,.\,BA+\sigma\,.\,BA\,.\,\frac{BA}{2}=\rho\,.\,AD\,.\,\frac{AD}{2}.$$

Whence $$AD^2=CE\,(3BE+BC),$$

and $$AD^2:AB^2=3\,\frac{\sigma}{\rho}-2+2\sqrt{1-\frac{\sigma}{\rho}}.$$

This being less than unity

$$\sigma:\rho<5:9.$$

Instead of introducing the stress at B, we might have taken moments about A for the equilibrium of the system, and about B for the rod BC.

33. Let the side AC be divided in the ratio $x:1-x$, by the surface of the fluid in the first position.

Then $$\sigma=(1-x)^2.$$

In the second position the area immersed is $\overline{1-x^2}$ times that of the triangle.

$\therefore$ The pressure on the hinge $=$ weight of displaced fluid $-\ W$

$$=W\left(\frac{1-x^2}{(1-x)^2}-1\right)$$

$$=2W\,\frac{1-\sqrt{\sigma}}{\sqrt{\sigma}}.$$

34. It is only necessary that the upward pressure due to these two liquids should be unaltered in amount and line of action.

This requires that the c. g. of these two portions should be in a vertical line and that their sp. grs. should be such that their combined weights may be equal to that of the water which is to replace one of them.

If the sp. gr. of the solid be alterable, the latter condition need not be satisfied, since a reduction or increase of its total weight will effect the necessary adjustment of the balancing forces.

Let the fraction x of the iron float in the lighter, and therefore $1-x$ in the heavier liquid. We have then

$$\cdot 8x + 13\cdot 6\,(1-x) = 7\cdot 8,$$
$$\therefore\ 5\cdot 8 = 12\cdot 8x,$$
$$x = \tfrac{29}{64}.$$

If s be the required sp. gr.,

$$0\,.\,x + 1 - x = s,$$
$$\therefore\ s = 1 - x = \tfrac{35}{64}.$$

Ratio of parts immersed $= 29 : 35$.

35. The weight of fluid displaced by half the cone is equal to its own weight, and the c. g. of the displaced fluid is vertically below that of the cone.

Let r be the radius of the base, $2a$ the vertical angle of the cone.

The whole pressure on the curved surface immersed is

$$w\,.\,\frac{1}{2}\pi r^2 \operatorname{cosec} a\,.\,\frac{4r}{3\pi} = \frac{2}{3}w\,.\,r^3 \operatorname{cosec} a.$$

Hence the c. g. of the curved surface is at a depth $\dfrac{4r}{3\pi}$.

The pressure on the half of the flat end which is immersed is

$$w\,.\,\frac{1}{2}\pi r^2\,.\,\frac{4r}{3\pi} = \frac{2}{3}wr^3.$$

36. Let r be the radius of the base.

The pressure on it is $\quad w\,.\,\pi r^2\,.\,r\cos a.$

The horizontal and vertical components of this are

$$w\,.\,\pi r^3\,.\,\cos^2 a,\quad w\,.\,\pi r^3 \sin a\,.\,\cos a.$$

Hence the horizontal and vertical components of the resultant pressure on the curved surface are

$$w\,.\,\pi r^3 \cos^2 a \text{ and } \tfrac{1}{3}w\,.\,\pi r^3 \cot a - w\,.\,\pi r^3 \sin a\,.\,\cos a.$$

$$\therefore\ \frac{\frac{1}{3}\cot a - \sin a\,.\,\cos a}{\cos^2 a}\,.\,\tan\theta = 1$$

or

$$(1 - 3\sin^2 a)\tan\theta = 3\sin a\,.\,\cos a.$$

37. The pressure on the vertical dividing plane is

$$w \cdot \pi r^2 \cdot r = w\pi r^3.$$

This is the horizontal component of the resultant pressure on half the curved surface.

All the pressures pass through the centre, and therefore the resultant does so likewise.

It acts by symmetry in a plane perpendicular to the dividi ng plane.

Its vertical component is $\frac{2}{3}w\pi r^3$.

$\therefore$ It makes an angle $\tan^{-1}\frac{3}{2}$ with the vertical.

Its magnitude is

$$\frac{\sqrt{13}}{3}w\pi r^3 = \frac{\sqrt{13}}{4} \times \text{weight of liquid in sphere.}$$

38. Using the result just obtained, the moment of the resultant pressure on one-half about the hinge is

$$\frac{\sqrt{13}}{3}w\pi r^3 \times r \cdot \frac{3}{\sqrt{13}} = w\pi r^4.$$

If T be the tension of the string

$$T \cdot 2r = w\pi r^4,$$

$$\therefore\ T = \tfrac{1}{2}w\pi r^3 = \tfrac{3}{8}\text{ weight of liquid.}$$

39. The pressure on the plane face is $w \cdot h'^2 \tan a \cdot h'/3 = \frac{1}{3}wh'^3 \tan a$ and acts at a point on the axis at depth $h'/2$.

The resultant vertical pressure on the curved surface is

$$w \cdot \tfrac{1}{6}\pi h'^3 \tan^2 a.$$

$\therefore$ Moment about hinge of pressure on immersed surface

$$= \frac{1}{6}w\pi h'^3 \tan^2 a \cdot \frac{h' \tan a}{\pi} + \frac{1}{3}wh'^3 \tan a \cdot \frac{h'}{2}$$

$$= \frac{1}{3}wh'^4 \tan a \cdot \sec^2 a.$$

If this exceed $\frac{1}{6}w\pi h'^3 \tan^2 a \cdot \frac{h \tan a}{\pi}$ the parts will not separate, this quantity being the moment of the weight of half the cone,

i.e. if $$h' \sec^2 a > h \tan^2 a$$

or $$h' > h \sin^2 a.$$

40. $(W-P)^2 + Q^2$ is the square of the resultant pressure on the circular base, which is constant so long as its centre remains unmoved.

41. Let h be the height of the cone, r the radius of the base.

The resultant vertical pressure on the horizontal section through the axis is whr^2.

The weight of the contained fluid

$$=\tfrac{1}{2}w \,.\, \tfrac{1}{3}h \,.\, \pi r^2.$$

$\therefore$ The resultant vertical pressure on the upper half of the curved surface

$$=(\text{weight of fluid in cone})\left(\frac{3}{\pi}-\frac{1}{2}\right).$$

42. Let a be the inclination of the axis to the vertical, h its length, A the sectional area, W the weight of the fluid displaced.

Difference of pressure on ends

$$=w \,.\, h\cos a \,.\, A = W\cos a.$$

Resultant horizontal pressure on the curved surface

$$= W\sin a\cos a.$$

$\therefore$ Resultant vertical pressure on the curved surface

$$= W\sin^2 a.$$

$\therefore$ The resultant is inclined to the vertical at an angle

$$\frac{\pi}{2}-a,$$

as is evident, since every part of it is perpendicular to the axis.

43. Volume not immersed : volume of cone

$$=1 : 2\sqrt{2}=1 : (\sqrt{2})^3.$$

Let ABC be a principal section of the cone, A the vertex, B being in the surface, ABC being an equilateral triangle.

Since the c. g.'s of the whole cone and of the unimmersed portion are in a vertical line, AC is vertical.

The volume of the unimmersed portion

$$=\frac{\sqrt{3}}{2\sqrt{2}} \cdot \frac{1}{\sqrt{3}} \times \text{that of the whole cone},$$

since the area of the section perpendicular to AC is

$$\frac{\sqrt{3}}{2\sqrt{2}} \text{ of the base.}$$

Hence the given condition is satisfied.

44. If θ be the semi-vertical angle, r the radius of the base, h the height,

$$\pi r^2 \cdot \frac{h}{4}=\frac{\pi r^2}{\sin\theta} \cdot \frac{h}{12},$$

$$\therefore\ \theta=\sin^{-1}\tfrac{1}{3}=\operatorname{cosec}^{-1} 3.$$

45. Let A be the area of the disc, r the radius of the sphere, σ its specific gravity, h the depth of water in the vessel.

Then $$Ah = \tfrac{4}{3}\pi r^3 (1-\sigma).$$

If the sphere be in contact with the disc and only just immersed, $h = 2r$, and the equation becomes

$$A = \tfrac{2}{3} \, . \, \pi r^2 (1-\sigma).$$

46. If θ is the inclination of the deck to the horizon

$$\tan\theta = \frac{5}{20 \times 21},$$ See Errata.

and the restorative moment of fluid pressure $= 9000\, h \sin\theta$, h being the metacentric height.

The moment of the 20 tons is $20 \times 21 \cos\theta$,

$$\therefore\ h = \tfrac{56}{25} = 2{\cdot}24 \text{ feet.}$$

CHAPTER V.

EXAMINATION.

1. $$\frac{C}{5} = \frac{F-32}{9} = \frac{R}{4}, \text{ and } F = 40^\circ,$$

$$\therefore R = 3^\circ\tfrac{5}{9}, \qquad C = 4^\circ\tfrac{4}{9}.$$

2. The required height $= \dfrac{13{\cdot}568}{5{\cdot}6} \times 30 = 72{\cdot}7$ inches nearly.

3. The pressure on each square inch is 1728 times as great as before. $\therefore$ the pressures on a side are as 1 : 12.

4. $$p = k\rho\,(1 + at).$$

The volume is reduced to one-eighth, and $\therefore$ the density is increased 8-fold. $\therefore$ the pressure is $8\,(1 + at)$ times as great. The surface of the smaller sphere is $\frac{1}{4}$ that of the larger,

$\therefore$ the pressure on it is $2\,(1 + at)$ times that on the larger.

5. A small aperture made at the highest point of a siphon would cause the liquid to flow out of each arm.

6. The column would be longer, as the vertical distance between the top and bottom of it must remain the same.

7. $$\frac{C}{5} = \frac{F-32}{9} = \therefore \frac{F+C-32}{14} = -\frac{32}{14} \because F + C = 0.$$

$$\therefore F = 11^\circ\tfrac{3}{7}, \qquad C = -11^\circ\tfrac{3}{7}.$$

8. It would be impossible to employ a siphon for the purpose of carrying fluids over heights exceeding five-sixths of those for which it can ordinarily be used at the sea-level.

9. (i) No effect, if the top of the siphon is less than 30 inches above the openings of the ends.

(ii) The mercury flows out of that end which is the lower.

10. The pressure is increased or decreased by an amount obtained by multiplying the area of the surfaces by the increase or decrease of the atmospheric pressure.

11. The wood, since it displaces more air and therefore experiences a greater upward pressure.

12. If the aperture be made in the longer branch above the level of the mercury in the shorter branch, the mercury above it will either pass to the closed end of the tube, or if the tube and hole be large, will run out.

If the hole be below the surface of the mercury in the shorter branch, some mercury will flow out till the level in the shorter branch is that of the aperture, unless the tube and aperture be large, in which case the mercury may not only fall a little in the longer arm, but may allow air to pass to the top of the tube and all that is above the aperture may run out.

If the hole be made in the shorter branch, the level falls in the longer branch, till its height above the hole is equal to what it was previously above the free surface, mercury meanwhile running out till the new surface is on a level with the aperture.

13. In each case to admit air above the surface of liquid in the vessel, so that it may more readily flow out.

14. Required force $= \pi \cdot 81 \cdot \frac{15}{2} = 1909$ lbs. nearly.

15. At the same level, since the weight of the glass is equal to that of the mercury displaced.

16. Not unless the top of the siphon is so near the greatest height over which it can carry the liquid for which it is being used, that the fall in the barometer reduces the possible height below the actual.

17. The weight of the displaced air being diminished, the tension of the string will be increased.

18. The volume is decreased to $\frac{4}{5}$ of its original value

$$\therefore \frac{p \times \frac{4}{5}}{273+16} = \frac{33\frac{1}{4} \times 1}{273+50}$$

$$p = \frac{5}{4} \times \frac{133}{4} \times \frac{289}{323}$$

$$= 37\tfrac{3}{16} \text{ inches of mercury.}$$

19. Let V be the volume of the body. W its weight, w the intrinsic weight of air when the barometric height is h. $\therefore \frac{h'}{h} w$ when the barometric height is h'.

Then
$$wV = \frac{1}{m} W$$

$$\frac{h}{h'} wV / W = \frac{h'}{mh},$$

which is the fraction of the weight lost when the barometric height is h'.

20. The liquid is of exactly one-half the sp. gr. of mercury and can therefore be carried over double the height by a siphon, i.e. over five feet.

21. The expansion is thirty-fold. Hence the pressure becomes one-thirtieth or $\frac{1}{3}$ ton wt. per sq. in.

22. At the end of the change of temperature the pressure has been increased in the ratio 1·3665 to 1.

The ratio of increase during the second rise of temperature is therefore

$$1{\cdot}3665 : 1{\cdot}03665$$

$$= 9110 : 6911.$$

23. If the volume of v litres at 0°, at the pressure due to 760 mm., change to V litres at $t°$ to h. mm.

$$\frac{v \times 760}{273} = \frac{V.h}{273+t}.$$

$$\therefore \quad v = V\frac{h}{760}(1+at) \text{ where } a = \frac{1}{273}.$$

The number of grammes is therefore

$$\frac{1{\cdot}293187}{760\,(1+at)}\,Vh = {\cdot}0017\ldots Vh/(1+at).$$

24. The pressure 10 feet below the surface exceeds that at the surface by $\frac{10 \times 30}{13{\cdot}5 \times 2{\cdot}5}$ inches of mercury

$$= 8\tfrac{8}{9} \text{ inches.}$$

$\therefore$ the volume becomes $\frac{350}{27} \times {\cdot}0001$ cub. in.

$$= {\cdot}000\dot{1}29\dot{6} \text{ cub. in.}$$

25. The density of water being 62·5, that of air is ·0013 × 62·5.

The pressure on a square foot is the weight of $2\frac{1}{2}$ cubic feet of mercury = 2·5 × 13·568 × 62·5 lbs. wt.

$$\therefore k = \frac{13{\cdot}568 \times 62{\cdot}5 \times 2{\cdot}5}{{\cdot}0013 \times 62{\cdot}5} = \frac{339200}{13}$$

$$= 26092 \text{ very nearly.}$$

EXAMPLES.

1. The volume is increased and $\therefore$ density diminished in the ratio $1 : n^3$.

$\therefore$ the pressure is altered in the ratio $n^3 : 1 + \alpha t$.

2. The pressure being constant, the density varies inversely as the absolute temperature.

3. If A is the area of the piston, W its weight, h the height of the cylinder, x the height of the piston in equilibrium, Π, Π' the pressures of the air outside and inside

$$\Pi h = \Pi' x, \text{ and } W = \Pi' A - \Pi A.$$

4. Let W be the weight, A the area of the piston, $2a$ the length of the cylinder, x the height above the base at which the piston is in equilibrium, ρ the density of the air when the piston is in the middle.

Then
$$\left[\frac{a}{x} - \frac{a}{2a-x}\right] k\rho A = W.$$

If we write
$$k\rho A = mW,$$

the equation becomes
$$x^2 - 2ax + 2ma(a - x) = 0.$$

5. The increase of pressure is $k\rho\alpha t$ per unit area.

6. The pressures are as $h : h'$, and therefore the densities of the air are in this ratio.

7. Let Π be the external air-pressure, W the weight of the piston, A its area, λ the modulus of elasticity, a the natural length of the string, t the increase of temperature, x the increase of length of the string.

Increase of pressure of air within

$$= \left[\frac{a}{a+x}(1 + \alpha t) - 1\right] \times \text{original pressure.}$$

Increase of tension of string $= \lambda \dfrac{x}{a}$.

And original pressure $= \Pi A + W$.

$$\therefore \frac{\lambda x}{a} = \frac{a\alpha t - x}{a + x}(\Pi A + W).$$

8. Let x be the fraction of the cylinder immersed when the air is admitted.

$$x + (1 - x)\cdot 0013 = \tfrac{3}{4}.$$

$$\therefore x = \tfrac{7487}{9987}.$$

The length above the surface is changed in the ratio $\cdot 25 : 1 - x$

$$= \cdot 9987 : 1.$$

9. (i) It rises so long as the density of the air remains less than that of water.

(ii) It falls.

10. Suppose the water to rise x feet.

$$15 : 15-x=33\tfrac{3}{4}+15-x : 33\tfrac{3}{4}.$$

$$\therefore\ 4x^2-255x+900=0.$$

The lesser root is $x=3\frac{3}{4}$.

The greater root is 60, which cannot apply to the present problem.

11. The hexagonal base contains six equilateral triangles, each one-fourth that into which the hexagon is deformed.

$\therefore$ The pressure is increased in the ratio 6 : 4, i.e. 3 : 2.

12. The pressure must be increased to 8 times its former value, i.e. the glass must be immersed to a depth $7h$, where h is the height of the water barometer.

13. The increased pressure causes the air in the envelope to decrease in volume.

14. Let a be the depth to which the original open surface of the mercury is lowered, w, w' the intrinsic weights of water and mercury, k, K the sectional areas of the tube.

If x be the increase of height of the mercury, x is made up of a rise of $Kx/(K+k)$ in the tube, and a fall of $kx/(K+k)$ in the reservoir.

$$\therefore\ w\left[a+\frac{kx}{K+k}\right]=w'x.$$

$$\therefore\ x=a\frac{w\left(1+\dfrac{K}{k}\right)}{w'\left(1+\dfrac{K}{k}\right)-w}.$$

If W be the weight of barometer and tube, W' the weight of an equal volume of water,

Tension of string $=W-W'+\dfrac{wxk}{K+k}$.

If a be increased, x is increased, $\therefore$ tension is increased.

15. Let x be the distance of the piston from its former position ρ the density of the atmosphere.

Then $$\frac{a}{a-x}\rho-\frac{a}{a+x}\rho=\rho\cos a.$$

Whence $$x=a\left[(1+\sec^2 a)^{\frac{1}{2}}-\sec a\right].$$

16. Let a be the length of the cylinder, b the length of it originally immersed, x the height of the water, Π the atmospheric pressure, h the height of the water-barometer.

$$\frac{a-b}{a-x}\Pi+\frac{x}{h}\Pi=\Pi.$$

$$\therefore\ x^2-(a+h)x+bh=0$$

determines the height x.

17. Let a be the length of each barometer, x, y the lengths of the tubes occupied by the air under a pressure equal to that of a length l of mercury.

$$h+\frac{x}{a-h}l=k+\frac{y}{a-k}l,$$

$$h'+\frac{x}{a-h'}l=k'+\frac{y}{a-k'}l,$$

$$\therefore\ x:y=\frac{h-k}{a-k'}-\frac{h'-k'}{a-k}:\frac{h-k}{a-h'}-\frac{h'-k'}{a-h},$$

supposing the temperature the same at the two observations.

18. Let x cubic inches be the required volume.

The pressure is

$$\frac{32-5+\frac{x}{12}}{32}\times \text{atmospheric pressure.}$$

$$\therefore\ \frac{20\times 32}{273+87}=\frac{\left(32-5+\frac{x}{12}\right)x}{273+15}.$$

Whence $x=17{\cdot}96\ldots$ cubic inches.

19. (i) If $2a$ be the edge of the cube, the radius of the sphere is $\sqrt{3}\,.\,a$.

$\therefore$ The volume is decreased from $4\sqrt{3}\pi a^3$ to $8a^3$.

The pressure is increased from p (say) to $\frac{\pi\sqrt{3}}{2}p$.

The areas are $12\pi a^2$ and $24a^2$ respectively.

$\therefore$ The whole pressures are $12\pi a^2 p$ and $12\pi a^2 p\,.\sqrt{3}$

i.e. as 1 to $\sqrt{3}$.

(ii) The volume is decreased from $8a^3$ to $\frac{4\pi}{3}a^3$.

The pressure is increased in the ratio $\pi : 6$.

The surfaces are $24a^2$ and $4\pi a^2$ respectively, and therefore the whole pressures are equal.

20. Let the pressure be ϖ when the length a of the cylinder is occupied. It will be $\frac{a}{b}\varpi$ when the length b is occupied.

The areas of surface are ap, bp, if p be the perimeter.

$\therefore$ The whole pressure is ϖap in each case, i.e. it is unaltered by moving the disc.

21. The formula $$z = k \log \frac{h}{h'},$$

gives $$2700 = k \log \frac{30}{27} = k \log \frac{10}{9}.$$

If x feet be the altitude corresponding to the height 21·87 of the barometer

$$x = k \log \frac{30}{21 \cdot 87} = k \log \left(\frac{10}{9}\right)^3$$

$$= 3k \log \frac{10}{9}$$

$$= 8100 \text{ feet.}$$

22. If a cubic centimetre be the unit of length, the density of water is unity. $\therefore$ that of air is ·00129.

The pressure on one square centimetre = the weight of 75·9 cub. cm. of mercury

$$= 75 \cdot 9 \times 13 \cdot 596 \text{ grammes weight}$$

$$= 1031 \cdot 9364 \text{ grammes weight}$$

$$= \cdot 00128992 \ldots \times \text{weight of 800 kilogrammes,}$$

which is almost exactly ·00129 the numerical value of the density.

23. The absolute zero of Fahrenheit's scale is $-459^\circ \cdot 4$.

The volume of 1 lb. of air is $\frac{1728}{\cdot 0763}$ cub. in. when the absolute temperature is $521^\circ \cdot 4$ and the pressure per sq. in. is the weight of 30 cub. in. of mercury $= \frac{30 \times 13 \cdot 596}{1728} \times 62 \cdot 5$ lbs. wt.

$$\therefore R = \frac{30 \times 13 \cdot 596 \times 62 \cdot 5 \times 1728}{1728 \times \cdot 0763 \times 521 \cdot 4}$$

$$= \frac{300 \times 13596 \times 625}{763 \times 5214} = 640 \cdot 8 \text{ nearly.}$$

24. Let $29 + x$ inches be the length of the tube.

Then $x - \cdot 4$ measures the volume of the air under a pressure ·4 inches of mercury.

And $x - \cdot 9$ measures the volume under a pressure $\cdot 5$ inches of mercury.

$$\therefore \cdot 4x - \cdot 16 = \cdot 5x - \cdot 45$$

$$2\cdot 9 = x.$$

The pressure of the air when $2\cdot 9$ inches is occupied by it is

$$\frac{\cdot 4\,(x - \cdot 4)}{2\cdot 9} = \frac{10}{29} = \cdot 3448\ldots \text{ inches.}$$

$\therefore$ the true reading when the barometer stands at 29 inches is $29\cdot 345$ very nearly.

25. The pressure at all points of the same horizontal plane must be the same. Hence points *below* the surface of the water where the barometer stands lowest will be on the same level as those *in* the surface where the barometer is highest.

In 50 miles the barometer varies $\cdot 05$ inch which corresponds to a pressure of $\cdot 6784$ inch of water, which is therefore the greatest rise above the mean level.

26. We have $$C = \lambda \,.\, B = \lambda^2 \,.\, A.$$

Let x, y be the lengths of the parts of the tube of area A, B, and let z be the length of the upper part filled with mercury when the middle part is just filled with glycerine.

The equivalent height of the mercury barometer is

$$x + \mu y + z.$$

If z increase to $z + u$, a volume Cu of glycerine rises out of the middle part and therefore an equal volume of mercury enters it, filling a length $\frac{C}{B}\,u = \lambda u$.

$\therefore$ Equivalent height of mercury barometer is now

$$x + \lambda u + \mu\,(y + u - \lambda u) + z.$$

The change in height of an ordinary barometer is thus

$$(\lambda + \mu - \lambda\mu)\,u$$

$\therefore$ the sensitiveness is greater in the ratio

$$1 : \lambda + \mu - \lambda\mu.$$

A similar calculation proves the same for a fall of the barometer.

CHAPTER VI.

EXAMINATION.

1. THE pressure is increased threefold approximately.

$\therefore$ the air occupies one-third of its former volume.

2. The air will flow out, for its pressure is equal to the pressure of the water within the bell, i.e. is greater than that of the water at the top of the bell.

3. To a height not exceeding that of the mercurial barometer.

4. Area of piston $A = 64$ times that of the plunger C.

Pressure on plunger $= 4 \times 2 = 8$ lbs.

$\therefore$ pressure on piston $= 64 \times 8 = 512$ lbs. wt.

5. The density is diminished by each stroke in the ratio 5 : 4.

Now $\left(\frac{4}{5}\right)^3 = \frac{64}{125}$, i.e. slightly exceeds $\frac{1}{2}$

and $\left(\frac{4}{5}\right)^4 = \frac{256}{625}$, i.e. is considerably less than $\frac{1}{2}$.

$\therefore$ the density is diminished one-half early in the fourth stroke.

7. The manometer has to register a pressure exceeding that of the atmosphere by three times.

$\therefore$ its length must be not less than $\frac{3}{2} \times 30 = 45$ inches.

[$\Pi' - \Pi = 2wx$, see Art. 117.]

8. Weight of water discharged per minute

$$= \pi \cdot \frac{1}{4} \cdot \frac{5}{2} \cdot 8 \cdot \frac{1000}{16} = \frac{5000\pi}{16} \text{ lbs.}$$

9. Weight of water discharged per minute

$$=\frac{\pi}{4}\cdot\frac{3}{2}\cdot 8\cdot\frac{1000}{16}=\frac{3000\pi}{16}\text{ lbs.},$$

since the effective range of the piston is only

$$33-31\tfrac{1}{2}=\tfrac{3}{2}\text{ feet.}$$

10. The exhaustion is accompanied by cooling which is sometimes sufficient to cause condensation of the vapour.

11. The density has become $\left(\frac{4}{5}\right)^5=\frac{1024}{3125}$ of its original amount.

$\therefore\ \frac{2101}{3125}=\cdot 67232$ of the air has been pumped out.

12. The increased pressure, due to the escape of carbonic acid gas, would drive out some of the water and $\therefore$ the displacement being increased without increase of weight, the tension of the rope would be diminished.

13. Let x feet be occupied by air.

$$\frac{x}{5}=\frac{33}{52+x},$$

$$x^2+52x-165=0,$$

$$(x+55)(x-3)=0,$$

$x=3$ as the negative root cannot apply to this problem.

14. The float rises out of the water as the water rises higher in the bell, owing to the increase of density of the air displaced by the upper part of it.

Examples.

1. Weight of water displaced $=\frac{P-P'}{W}\times W=P-P'$,

$\therefore$ the bell is in equilibrium, and as in Art. 184 this equilibrium is unstable.

2. After three strokes, the height of the mercury

$$=\left(\frac{3}{4}\right)^3\times 30=12\tfrac{2}{3}\text{ inches nearly.}$$

3. Let h be the height of the barometer in inches, x the rise after four strokes.

The pressure in the tube after three strokes = that due to a column $\frac{3h}{2}$ of mercury. $\therefore$ pressure in receiver is due to the column $\frac{3h}{2}+5$;

but $$\frac{\rho_3}{\rho}=1+3\frac{B}{A}, \quad \therefore \frac{B}{A}=\frac{1}{6}+\frac{5}{3h}.$$

$$\therefore \frac{\rho_4}{\rho}=1+4\frac{B}{A}=\frac{5}{3}+\frac{20}{3h}.$$

$$\therefore \frac{\frac{5h}{3}+\frac{20}{3}-x}{h}=\frac{15}{15-x} \text{ or } \frac{15}{15-x}+\frac{x}{h}=\frac{5}{3}+\frac{20}{3h}.$$

If $h=30$ inches, $x=6{\cdot}1$ inches nearly.

4. Let A be the sectional area of the bell, and suppose the air to occupy $x+u$ feet when the bucket has been drawn up.

The pressure is now increased from $h+a+x$ to $h+a+x+u$.

$\therefore$ The volume is decreased to

$$\frac{h+a+x}{h+a+x+u}Ax=Ax\left(1-\frac{u}{h+a+x}\right) \text{ q. p.}$$

$\therefore$ Volume of water is

$$Au\left(1+\frac{x}{h+a+x}\right)=Au\frac{h+a+2x}{h+a+x}.$$

The increased displacement is Au, the weight of which is

$$W.\frac{h+a+x}{h+a+2x}.$$

This is the increase of upward pressure, but there is an additional weight W.

$\therefore$ The tension of the rope increases by

$$\frac{Wx}{h+a+2x}.$$

5. The level of the water rises in the bell, and there is therefore less displacement, so that the tension is from this cause increased. On the other hand the tension is diminished by the weight of the man, who is no longer supported by the bell.

If we consider the average intrinsic weight of the man to be equal to that of water, it will be seen that the tension is diminished.

6. h being the height of the water-barometer,

$\frac{m}{m-1}h$ is the depth to which the bell is sunk.

At the end of one second the depth is

$$\frac{m}{m-1}h+n \text{ feet,}$$

$\therefore$ the amount of air to be pumped in

$$=\frac{\frac{m}{m-1}h+n}{\frac{m}{m-1}h}V-V=\left(1-\frac{1}{m}\right)\frac{n}{h}V.$$

7. Let the length of the stroke be x feet.

$$\frac{4x}{10}=\frac{33}{23}.$$

$$\therefore\ x=3\tfrac{27}{46}=3 \text{ feet } 7 \text{ inches about.}$$

8. Let x be the depth to which the bell is immersed, y the depth of water in it.

$$\sigma h+\rho(x-y)=\sigma h'.$$

And
$$\frac{a}{a-y}=\frac{h'}{h};$$

hence
$$x=\frac{\sigma}{\rho}(h'-h)+a\frac{h'-h}{h'}=\left(\frac{\sigma}{\rho}+\frac{a}{h'}\right)(h'-h).$$

9. Let a be the length originally occupied by air,

h the height of the barometer,

ρ the density of the atmosphere,

ρ_n the density in the receiver after n strokes of the piston,

x the difference of height required.

$a+\frac{x}{2}$ is the length now occupied by air,

$$\therefore\ \frac{ah}{a+\frac{x}{2}}-x=\frac{\rho_n}{\rho}h$$

or
$$x^2+2a(x-h)+\frac{\rho_n}{\rho}h(2a+x)=0.$$

10. Let V, v be the volumes of the receiver and barrel, A the area, w the weight of the valve, Π the atmospheric pressure. The pressure in the barrel when the fraction x of the nth descent remains to be accomplished is $\frac{1}{x}\left(\frac{V}{V+v}\right)^{n-1}\Pi$.

If the valve begins to open at this point, this pressure $=w/A$;

$$\therefore\ x=\left(\frac{V}{V+v}\right)^{n-1}\frac{\Pi A}{w}.$$

11. Let σ be the density in the receiver at any time.

If
$$\sigma\frac{h+\beta}{\beta} \not> \rho\frac{a}{a+h}$$
the two parts of the barrel will never communicate;

$\therefore$ the limiting density is $\rho \cdot \dfrac{a\beta}{(a+h)(\beta+h)}$.

12. The density after n strokes of the piston being ρ_n, let the fraction x of the stroke be completed when the valve opens. Then
$$\rho_n = \left(\frac{A}{A+B}\right)^n \rho;$$
$$B(1-x)\rho = B\rho_n;$$
$$1-x = \frac{\rho_n}{\rho} = \left(\frac{A}{A+B}\right)^n.$$

The pressure below the piston
$$= \frac{A}{A+Bx}\frac{\rho_n}{\rho} \times \text{atmospheric pressure.}$$

$\therefore$ tension of piston rod : atmospheric pressure on piston
$$= 1 - \frac{A}{A+Bx} \cdot \left(\frac{A}{A+B}\right)^n : 1$$
$$= 1 - \left(\frac{A}{A+B}\right)^n : 1 - \left(\frac{A}{A+B}\right)^n \cdot \frac{B}{A+B}.$$

13. We must have $\dfrac{\frac{\Pi v}{v'}}{t} = \dfrac{\frac{\Pi v}{v'}+y}{t+x}$;

since the pressure within the bell is $\dfrac{\Pi v}{v'}$ and v' is unchanged,
$$\therefore \frac{x}{t} = \frac{yv'}{\Pi v}.$$

14. The pressure $= \dfrac{AB}{BC} \times$ atmospheric pressure.

Increase of pressure $= \dfrac{AC}{CB} \times$ atmospheric pressure.

$\therefore$ if the pressure increases uniformly,

$AC : CB$ increases uniformly.

15. The pressure is doubled by 20 strokes of the condenser. It is then decreased in the ratio $\left(\dfrac{20}{21}\right)^{14}$ by 14 strokes of the pump.

Now
$$\left(\frac{20}{21}\right)^{14} = \cdot 50506\ldots = \text{about } \tfrac{1}{2}.$$
i.e. the density is approximately restored to its original value.

16. The density increases in A. P. after each stroke.

$\therefore$ The volume of the air in the condenser, and hence its length, decreases in H. P.

Let ρ_p be the density in the condenser after p strokes.

In the next stroke a volume V at density ρ_p and a volume $v+x$ at density ρ are condensed into a space $V+v$.

$$\therefore \rho_{p+1}(V+v)=\rho(v+x)+\rho_p V,$$

$$\rho_{p+1}\left(\frac{V+v}{V}\right)^{p+1}-\rho_p\left(\frac{V+v}{V}\right)^p=\rho\frac{v+x}{V}\left(\frac{V+v}{V}\right)^p.$$

$$\therefore \rho_n\left(\frac{V+v}{V}\right)^n-\rho=\rho\frac{v+x}{V}\left[\frac{\left(\frac{V+v}{V}\right)^n-1}{\frac{V+v}{V}-1}\right]$$

$$=\rho\frac{v+x}{v}\left[\left(\frac{V+v}{V}\right)^n-1\right],$$

$$\rho_n=\rho\left[\frac{v+x}{v}-\frac{x}{v}\left(\frac{V}{V+v}\right)^n\right].$$

And the pressure is

$$\Pi\left[\frac{v+x}{v}-\frac{x}{v}\left(\frac{V}{V+v}\right)^n\right].$$

CHAPTER VII.

EXAMINATION.

1. THE mass of water displaced by the solid must be 7 lbs.;

$\therefore$ its sp. gr. is to that of water as 5 : 7.

2. The mass of A displaced by the solid is 9 lbs., and the mass of B displaced is 7 lbs.

$\therefore$ the sp. gr.'s of A and B are as 9 : 7.

3. $\left(5 - \frac{\pi}{256}\right)s_1 = \left(5 - \frac{2\pi}{256}\right)s_2$.

4. If V is the volume of cork required and V' the volume of the iron,
$$V' \times 7{\cdot}6 \times \frac{1000}{16} = 6,$$
and
$$V \times {\cdot}24 \times \frac{1000}{16} + 6 = (V' + V)\frac{1000}{16}; \therefore V = \frac{198}{1805}.$$

5. Equal volumes of the body, water, and spirit weigh 250, 210 and 200 grains respectively.

$\therefore$ the sp. gr.'s of the body and the spirit are $\frac{25}{21}$ and $\frac{20}{21}$.

6. If k is the sectional area of the stem of the hydrometer, and if s is the sp. gr. of the liquid,
$$(V - ka)(62{\cdot}5) = W, \text{ and } (V - ka + v')(62{\cdot}5)s = W + w.$$
$$\therefore s(W + 62{\cdot}5v') = W + w.$$

7. The weights of water and spirit displaced are respectively equal to the weights of 200 grains and of 160 grains;

the sp. gr. is therefore $\frac{4}{5}$.

8. The weight of water displaced by the wood is equal to the weight of 25 oz.; and the weight of the metal in vacuum is equal to the weight of 20 oz.;

$\therefore$ the sp. gr. of the wood is $\frac{4}{5}$.

9. Whole weight of water displaced is equal to the weight of

$$(57+42-38)\text{ lbs., i.e. of }61\text{ lbs.}$$

The silver displaces the weight of 4 lbs;

$\therefore$ the wood displaces the weight of 57 lbs., and its sp. gr. = that of water.

EXAMPLES.

1. If w is the weight of the material, $4w$ is the weight of the sinker in water;

$\therefore$ $2w$ is the weight of water displaced by the material, and its sp. gr. is ·5.

2. The volume of the metal $= 1 - \left(\frac{8}{9}\right)^3$ cubic inches.

Its weight = that of $1 + \frac{4·34}{2}$, or 3·17 cubic inches of water.

$\therefore$ its sp. gr. $= \frac{729}{217} \times 3·17 = 10·65$ nearly.

3. There are $8·22 - 6·3$ or 1·92 grains of wax.

The mass of a volume of water equal to the volume of the wax is therefore 2 grains.

The displacement of water by the two is $8·22 - 3·02$, or 5·2 grains.

$\therefore$ the mass of a volume of water equal to the volume of the salt is 3·2, and the sp. gr. is $\frac{6·3}{3·2}$ or 1·96875.

4. The ratio is $7 : 7\frac{1}{2}$ or $14 : 5$.

5. A quantity of liquid equal in volume to the solid weighs $1\frac{1}{2}$ oz.

$\therefore$ sp. gr. of solid : that of liquid :: 4 : 3.

6. Let w_1, w_2, w_3 be the weights of the gold, the diamond, and a ruby respectively; then, taking the weight of a grain as the unit of weight,

$$w_1 + w_2 + 2w_3 = 44\tfrac{1}{4}.$$

$$\frac{w_1}{16\frac{1}{2}} + \frac{w_2}{3\frac{1}{2}} + \frac{2w_3}{3} = 5\tfrac{1}{2}.$$

The weight of a ruby in water is that of two grains; $\therefore$ in air it is that of 3 grains.

Hence $$w_2 = 5\tfrac{1}{4}.$$

7. Let x gallons of water be mixed with one of whiskey,

$$\cdot 75 + x = \cdot 8\,(1+x), \ \therefore\ x = \cdot 25,$$

and price should be $\frac{4}{5}$ of 16, or $12\frac{4}{5}$ shillings per gallon.

8. Let V be the original volume of the instrument, x the volume not immersed in water, $x+u$ the volume not immersed in liquid of sp. gr. 1·002

$$V - x = (V - x - u)\,1{\cdot}002. \quad \text{See errata}$$

$$501u = V - x.$$

The mass was originally $\rho\,(V-x)$.

It is now $\rho\,(V-x-u)$,

ρ being the density of water.

$\therefore$ the fraction $\dfrac{u}{V-x} = \dfrac{1}{501}$ has been knocked off.

9. Let V be the volume of the hydrometer, a the sectional area of the stem.

$$(V - aa)\,\rho_1 = (V - ba)\,\rho_2 = (V - ca)\,\rho_3 = M,$$

the mass of the hydrometer.

$$\therefore\ V - aa = \frac{M}{\rho_1},$$

$$V - ba = \frac{M}{\rho_2},$$

$$V - ca = \frac{M}{\rho_3}.$$

$$\therefore\ (b-c)\,a = M\,\frac{\rho_2 - \rho_3}{\rho_2\rho_3}.$$

$$\therefore\ \frac{b-c}{\rho_1} + \frac{c-a}{\rho_2} + \frac{a-b}{\rho_3} = 0.$$

10. If ρ_1, ρ_2, ρ_3 are the intrinsic weights of the liquids, V the volume and W the weight of the common hydrometer, $\lambda_1, \lambda_2, \lambda_3$ the lengths of its stem above the surfaces,

$$W = \rho_1\,(V - \kappa\lambda_1) = \rho_1\kappa l_1 = \rho_2\kappa l_2 = \rho_3\kappa l_3.$$

If V' is the volume immersed and W' the weight of the Nicholson's hydrometer,

$$w_1 + W' = \rho_1 V',\ w_2 + W' = \rho_2 V',\ w_3 + W' = \rho_3 V',$$

whence $$w_2 - w_3 = V\,(\rho_2 - \rho_3) = \frac{VW}{\kappa}\,\frac{l_3 - l_2}{l_2 l_3}, \text{ \&c.}$$

and the stated result follows at once.

11. If W is the real weight of the body, and w, w' the real weights of the weighing pieces, we have

$$\frac{W}{\rho}(\rho-1)=\frac{w}{\rho'}(\rho'-1),$$

and
$$\frac{W}{\rho}\left(\rho-\frac{29}{30}\right)=\frac{w'}{\rho'}\left(\rho'-\frac{29}{30}\right).$$

The apparent weights of the body are w and w',

and since
$$\frac{1-\dfrac{1}{\rho}}{1-\dfrac{29}{30\rho}}=\frac{w}{w'}\cdot\frac{1-\dfrac{1}{\rho'}}{1-\dfrac{29}{30\rho'}},$$

it follows that
$$\frac{w'-w}{w}=\frac{\rho'-\rho}{(\rho-1)(30\rho'-29)}.$$

This is an increase or a decrease according as $\rho' >$ or $< \rho$.

12. Let W be the weight of the bottle, V its internal volume, v the volume of its material.

w_a, w_b, w_c the intrinsic weights of A, B, C.

$$W+Vw_a-(V+v)\,w_b=A_b,$$
$$W+Vw_a-(V+v)\,w_c=A_c,$$
$$W+Vw_b-(V+v)\,w_c=B_c,$$
$$W+Vw_b-(V+v)\,w_a=B_a,$$
$$W+Vw_c-(V+v)\,w_a=C_a,$$
$$W+Vw_c-(V+v)\,w_b=C_b,$$

$\therefore\ A_b+B_c+C_a=A_c+B_a+C_b.$

CHAPTER VIII.

EXAMINATION.

1. PRESSURE of mixture $=\dfrac{1728\times 15+1\times 60}{1729}=15{\cdot}026$ lbs. wt. per sq. inch.

3. Dew would be deposited on the furniture, &c., in the room.

8. Temperature of mixture $=\dfrac{3\times 45+6\times 90}{9}=75^\circ$.

9. The dryness of the air, and its capacity to absorb moisture owing to its low pressure, causes evaporation from the surface of the skin to go on more rapidly than moisture can be supplied from the subjacent tissues.

10. When the volume is $V+V'$ and the temperature t, the pressure is

$$\frac{pV+p'V'}{V+V'}.$$

When the volume is changed to U and the temperature to t', the pressure becomes

$$\frac{273+t'}{U}\cdot\frac{pV+p'V'}{273+t}$$

$$=\frac{pV+p'V'}{U}(1+a\overline{t'-t})\text{ where }a=\tfrac{1}{273}.$$

11. The new pressures being ϖ, ϖ', and τ being the increase of temperature, (taking t as absolute temperature)

$$\frac{p}{t}=\frac{\varpi}{t+\tau},\qquad \therefore\ \varpi-p=\frac{p\tau}{t};$$

$$\frac{p'}{t}=\frac{\varpi'}{t+\tau},\qquad \therefore\ \varpi'-p'=\frac{p'\tau}{t}.$$

$\therefore$ the greater increase of pressure takes place where the original pressure was the greater.

Pressure at temperature zero $= \frac{p+p'}{2} \cdot \frac{273}{t}$;

or if t represent temperature on the centigrade scale,

$$\frac{p+p'}{2}(1-at).$$

EXAMPLES.

1. If $\theta°$ be the final temperature,

$$2(100-\theta)\times \cdot 12 = (5+16\times \cdot 2)(\theta - 20)$$
$$\therefore\ \theta = 22° \tfrac{58}{211}.$$

2. In this case, with same notation,

$$\cdot 12(120-\theta) + 2\times \cdot 1(90-\theta) = (6+10\times \cdot 2)(\theta-10);$$
$$\therefore\ \theta = 13° \tfrac{58}{104}.$$

3. The space being saturated with vapour, one-half of it is condensed when the compression takes place.

Hence while the pressure of the air is doubled, that of the vapour remains unaltered.

Hence the difference of the observed pressures is the required original pressure of the air free from vapour.

4. The air in the ball-room was saturated with moisture in the form of vapour. This being suddenly frozen, became snow, when the temperature of the room was suddenly lowered below freezing-point.

5. The pressures (reduced to zero) in the air-space, are as

$$E_1(1-et_1) : E_2(1-et_2);$$

$\therefore$ a being the required ratio,

$$\frac{a^{t_1}}{a^{t_2}} = \frac{E_1(1-et_1)}{E_2(1-et_2)};$$

or
$$a = \left[\frac{E_1(1-et_1)}{E_2(1-et_2)}\right]^{\frac{1}{t_1-t_2}}.$$

6. The pressure on the under side of the piston never rises beyond twice its original value.

That above the piston is $\frac{3}{4}$ its original value.

$\therefore$ the weight of the piston $= 2 - \frac{3}{4} = \frac{5}{4}$ original pressure on it.

7. The vapour in the space above the water being condensed, the temperature of the water is above its boiling-point at the reduced pressure.

8. The freezing-point is lowered $\cdot 3^\circ$ $\therefore$ $\cdot 15$ of the heat required to raise the whole mass (if melted) through 1° is set free and expended in melting a portion of the mass.

$\therefore$ the mass melted is $\dfrac{\cdot 15}{79}$ = rather less than $\frac{1}{500}$th of the whole.

Or thus, the lowering of temperature $=(\cdot 0075)\cdot 40=\cdot 3$; so that $-\cdot 3^\circ$ is the final temperature.

Let the portion x of the mass of ice M be melted; then

$$\begin{aligned}79x &= \text{heat taken out of } x \text{ of water}\\ &\quad + \text{heat taken out of } (M-x) \text{ of ice}\\ &= x(\cdot 3)+\tfrac{1}{2}(M-x)(\cdot 3),\end{aligned}$$

taking the specific heat of water to be unity.

From this equation,

$$\frac{x}{M}=\frac{\cdot 3}{158-\cdot 3}=\frac{3}{1577}=\frac{1}{526} \text{ nearly.}$$

9. Let θ° be the resulting temperature

$$(\theta-40)\cdot 06=10(100-\theta)$$

$$\theta=99^\circ\,\tfrac{323}{503}.$$

Heat absorbed by silver = heat given out by water

$$=\tfrac{10}{16}\cdot\tfrac{180}{503}=\tfrac{225}{1006} \text{ units.}$$

The work equivalent of this is

$$\frac{225\times 772}{1006} \text{ foot-pounds,}$$

which would raise the ounce of silver to a height

$$16\times 172\tfrac{2}{3} \text{ feet}=\frac{16\times 518}{9} \text{ yards}=921 \text{ yards nearly.}$$

10. Let p, ϖ, be the pressures of the air and vapour.

$$p+\varpi=\Pi.$$

On compression the air-pressure becomes np, but the pressure of the vapour can never exceed ϖ, for condensation takes place.

$$\therefore\ np+\varpi=\Pi'.$$

$$\therefore\ \varpi=\frac{n\Pi-\Pi'}{n-1}$$

$$p=\frac{\Pi'-\Pi}{n-1}.$$

11. Let r be the radius of the cylinder, W its weight, ϖ the pressure of the vapour, Π that of the atmosphere.

If $(3\varpi - \Pi)\pi r^2 < W$, the vapour will be condensed

$$\text{i.e. } W \text{ must not exceed } (3\varpi - \Pi)\pi r^2.$$

12. The temperature of the ice is raised to 32°, the ice is then melted, and as the final temperature is 59°, the temperature of the water is increased from 32° to 59°, i.e., it is raised 27°.

If v is the volume of the ice, the mass of the ice is $\cdot 96v \times 62\cdot 5$, and the number of units of heat required for these three changes is therefore

$$\cdot 96v \times 62\cdot 5\,\{\cdot 5 \times 2 + 144 + 27\}.$$

Again, the air being lowered 1° of temperature, its loss of heat, if nv represent its volume, is

$$\cdot 0013nv \times 62\cdot 5 \times \cdot 2375.$$

Equating this to the former expression, we find that

$$n = \frac{16512}{\cdot 030875} = 534801\cdot 6....$$

CHAPTER IX.

EXAMPLES.

1. THE greatest tension in each cylinder is at the base. We have $r^2h = r'^2h'$, $t = whr$, and $t' = wh'r'$;

$$\therefore\ t : t' = r' : r.$$

2. $t = pr$, and $2t = p' . 3r$,

$$\therefore\ p : p' = 3 : 2.$$

3. A bar one square inch in section can support 4000 lbs.; $\therefore$ the greatest value of t is 400 lbs., and the greatest value of p is 80 lbs.

4. If r, r' be the radii, e, e' the thicknesses,

$$r^2e = r'^2e';$$

and the strengths are as

$$\frac{2e\tau}{r} : \frac{2e'\tau}{r'} \text{ or as } r'^3 : r^3.$$

5. Let $\frac{4}{3}\pi x^3$ be the volume of air, at atmospheric pressure Π, which is forced in.

Pressure with radius $b = \dfrac{x^3 + a^3}{b^3}\Pi$;

and with radius $c \qquad = \dfrac{x^3 + a^3}{c^3}\Pi$.

Let t be the tension with radius b, and t' with radius c; then

$$\Pi\left(\frac{x^3 + a^3}{b^3} - 1\right) = \frac{2t}{b}, \qquad \Pi\frac{x^3 + a^3}{c^3} = \frac{2t'}{c},$$

and

$$t : t' = b^2 - a^2 : c^2 - a^2,$$

so that

$$\frac{x^3 + a^3 - b^3}{b^2} : \frac{x^3 + a^3}{c^3} :: b^2 - a^2 : c^2 - a^2,$$

which determines x.

6. The tension and pressure all along the band being the same, the curvature is the same, i.e. the band lies in a circular arc.

Let r be the radius of the circle at any time,

2α the angle subtended at the centre,

$2a$ the unstretched length of the band,

b the depth of the box, p the pressure inside;

$$a = r\sin\alpha; \quad t = (\Pi - p)\, r.$$

And
$$2r\alpha = 2a\left(1+\frac{t}{\lambda}\right) = 2a + 2ar\frac{\Pi - p}{\lambda};$$

$$\therefore\ \alpha - \sin\alpha = \alpha . \frac{\Pi - p}{\lambda}.$$

When the band touches the bottom of the box,

$$r(1-\cos\alpha) = b;$$

$$\therefore\ \alpha = 2\tan^{-1}\frac{b}{a};$$

and
$$\Pi - p = \frac{\lambda}{a}\left[2\tan^{-1}\frac{b}{a} - \frac{2ab}{a^2+b^2}\right].$$

If $b > 2a$, the arc will gradually become a semicircle, and then the ends will be flattened against the vertical sides of the box, the free portion forming a semicircle.

7. At a depth h below the surface, the tension is wha.

Let $2b$ be the length of the side of the box,

r the radius of the curved portions of the membrane.

$$2\pi r + 4(2b - 2r) = 2\pi a;$$

$$\therefore\ r = \frac{4b - \pi a}{4 - \pi};$$

$\therefore$ the tension at a depth $h = wh . \dfrac{4b - \pi a}{4 - \pi}$.

The change $= wh . \dfrac{4(a-b)}{4-\pi}$;

$\therefore$ change : original tension $= 4(a-b) : a(4-\pi)$.

8. If p, r be the increase of pressure and radius, a being the original radius, t the tension,

$$2\pi r = 2\pi a\frac{t}{\lambda}; \ \therefore\ t = \frac{\lambda r}{a}.$$

But
$$t = p(a+r);$$

$$\therefore\ r = \frac{pa^2}{\lambda - pa}.$$

If $r = a$, $\lambda = 2pa$.

9. This example belongs to Chap. XIV.

10. The pressure at the foot of the main

$$=300 \times 62\cdot 5 = 18750 \text{ lbs. wt. per sq. foot.}$$

If τ be the required thickness in inches,

$$5 \times 2240\, \tau = \frac{18750}{1728} \times \frac{9}{2};$$

$$\therefore\ \tau = \frac{1}{230} \text{ nearly.}$$

11. Let τ be the required thickness in inches

$$2000\, \tau = \frac{384 \times 62\cdot 5}{1728} \times 6;$$

$$\therefore\ \tau = \frac{1}{24} \text{ inch.}$$

12. If τ is the stress per unit of areal section, and if e and e' are the thicknesses,

$$\frac{e\tau}{R} = \frac{e'\tau}{r}, \quad \therefore\ e : e' = R : r.$$

13. Let h feet be the required height of water and let r be the radius of the cylinder.

Then $$\tau = \frac{144\, W}{A}, \quad t = \frac{\tau a}{12}, \quad \frac{t}{r} = 62\cdot 5 h.$$

$\therefore$ volume of water $= \pi r^2 h = 12 \pi r a\, W / 62\cdot 5 A$.

14. Let r be the radius of the circular portions at any time, a the original radius.

The length of the perimeter of the tube is

$$2\pi r + 8(a - r).$$

The tension $$= \lambda \frac{8 - 2\pi}{2\pi} \left(\frac{a - r}{a} \right).$$

The pressure is $\therefore\ \lambda \cdot \dfrac{8 - 2\pi}{2\pi r} \cdot \dfrac{a - r}{a}$,

which is proportional to $$\frac{8(a - r)}{2\pi r}$$

i.e. to the ratio of the straight and curved parts of the tube.

15. Let p be the final pressure within the envelope,

$$p\,.\,\tfrac{4}{3}\pi r'^3 = 2\Pi\,.\,\tfrac{4}{3}\pi r^3,$$

$$\therefore\ pr'^3 = 2\Pi r^3.$$

The tension at any point is $\frac{r'}{2}(p-\Pi)$

$$= \frac{\Pi}{2r'^2}(2r^3 - r'^3).$$

16. *Vide* 5.

17. The resultant of the tensions round a horizontal section is equal to the resultant vertical pressure on the portion of the cone above that section, i.e. between it and the vertex.

If h be the height of the cone, x the depth of the section, t the tension at the section,

$$2\pi x \tan a\,.\,t\,.\cos a = \tfrac{2}{3}w\pi x^3 \tan^2 a,$$

$$\therefore\ t = \tfrac{1}{3}w \tan a \sec a\,.\,x^2.$$

18. Let h be the height of the bag, $4a$ the parameter of the parabola, so that if r be the radius at a distance x from the vertex $r^2 = 4ax$.

t being the tension,

$$2\pi r\,.\,t\,.\frac{2x}{\sqrt{r^2+4x^2}} = w\pi r^2\,.\left(h - \frac{x}{2}\right);$$

$$\therefore\ t = \tfrac{1}{4}wr\,.\sqrt{\frac{(x+a)}{x}}(2h-x)$$

$$= \tfrac{1}{2}w\sqrt{a(x+a)}\,(2h-x).$$

At the vertex $x=0$,

$$\therefore\ t = wah,$$

or thus; radius of curvature at vertex $=2a$, and pressure there $=wh$,

$$\therefore\ \frac{2t}{2a} = wh, \text{ or } t = wah.$$

19. $$t\,.\,2\pi\,.\,PN\,.\frac{2AN}{\sqrt{PN^2+4AN^2}} = w\pi\,PN^2\,.\,\frac{AN}{2}.$$

$$\therefore\ t \propto \sqrt{AS+AN}\,.\,AN \propto AN\sqrt{SP}.$$

20. Let p, ϖ be the pressures of the air and vapour.

Then $p+\varpi=\Pi$ the atmospheric pressure.

Let a be the original radius.

When radius is doubled, the pressure within exceeds that without by $p+\varpi$.

The tension $=a(p+\varpi)$.

When radius is trebled, the pressure of the air is $\frac{77}{27}p$, and of the vapour ϖ.

$\therefore$ Inner pressure exceeds outer by $\frac{50}{27}p$.

$\therefore$ $\frac{25}{9}pa$ is the tension.

$\therefore$ $\frac{25}{9}ap : a(p+\varpi)=8 : 3$.

$$24(p+\varpi)=25p,$$

or

$$p=24\varpi$$

and

$$\varpi=\frac{1}{25}\Pi.$$

CHAPTER X.

EXAMPLES.

1. THE product of height and diameter of tube is constant.

For water it is given as ·048, for alcohol ·02.

∴ In a tube ·25 inch in diameter,

water rises ·192 inch, alcohol rises ·08 inch.

In a tube ·05 inch in diameter,

water rises ·96 inch, alcohol rises ·4 inch.

2. Depression $= \frac{\cdot 08 \times \cdot 15}{\cdot 025} = \cdot 048$ inch.

3. $p - \Pi = \frac{2t}{r} = \frac{36}{5}$, $\quad p' - \Pi = \frac{7}{8}$ in grains weight per square inch,

also
$$V\Pi = \frac{4}{3}\pi r^3 p, \text{ and } V'\Pi = \frac{4}{3}\pi r'^3 p,$$
$$\therefore\ V : V' = 7^3\left(\Pi + \frac{36}{5}\right) : 12^3\left(\Pi + \frac{7}{8}\right),$$

Π being the pressure of the atmosphere measured in grains weight per square inch.

4. Since $81 + 418 < 540$, it follows that equilibrium cannot exist, art. 167, page 160. The drop of water therefore spreads out over the mercury.

5. The reason in this case is the same.

6. The tension, τ, of the film being normal to the thread, it follows that the tension of the thread is constant, and, τ being constant, it follows that the curvature of the thread is constant.

7. Let t be the tension of the film, Π the atmospheric pressure.

The pressures inside the bulbs are
$$\Pi + \frac{2t}{r},\ \ \Pi + \frac{2t}{r'} \text{ and } \Pi + \frac{2t}{R},$$
$$\therefore\ r^3\left[\Pi + \frac{2t}{r}\right] + r'^3\left[\Pi + \frac{2t}{r'}\right] = R^3\left[\Pi + \frac{2t}{R}\right].$$
$$\therefore\ \Pi\,(R^3 - r^3 - r'^3) = 2t\,[r^2 + r'^2 - R^2].$$

8. Let t be the tension of the film,

Π the atmospheric pressure.

$$p_0 - \frac{2t}{r_0} = \Pi = p - \frac{2t}{r}.$$

$$p \cdot r^3 = \Pi \cdot a^3 + p_0 \cdot r_0^3.$$

$$2t = (p_0 - \Pi) r_0 = (p - \Pi) r.$$

$$\therefore \; p_0 r_0 = pr - \Pi (r - r_0).$$

$$\therefore \; pr^3 = \Pi a^3 + pr \cdot r_0^2 - \Pi (r - r_0) r_0^2.$$

$$p = \Pi \frac{a^3 - r_0^2 (r - r_0)}{r (r^2 - r_0^2)},$$

$$p_0 = \Pi \frac{a^3 - r^2 (r - r_0)}{r_0 (r^2 - r_0^2)}.$$

9. Take the wetted area between the plates of sensible extent, i.e. take B large compared with d, and let Π be the atmospheric pressure, and Π' the pressure of the liquid. Taking a small length e of B, which may be regarded as practically straight, we have

$$\Pi ed = \Pi' ed + 2te \cos a.$$

The plates are pressed together by two opposite forces, each equal to

$$(\Pi - \Pi') A + tB \cos a,$$

or,
$$A \frac{2t \cos a}{d} + Bt \sin a.$$

10. We have
$$m = \frac{4}{3} \pi \rho a^3.$$

Let t be the tension of the film,

$$k\rho - \frac{2t}{a} = \text{atmospheric pressure.}$$

$$= \frac{3mk}{4\pi a^3} - \frac{2t}{a}.$$

Let a become $a + x$.

This expression becomes (neglecting squares of x)

$$\frac{3mk}{4\pi a^3}\left[1 - 3\frac{x}{a}\right] - \frac{2t}{a}\left[1 - \frac{x}{a}\right].$$

The increase is
$$\frac{2x}{a^2}\left[t - \frac{9mk}{8\pi a^2}\right].$$

If this is to be always positive, x must be positive or negative according as $t >$ or $< 9mk/8\pi a^2$.

11. The upward forces on the cube when a volume V is immersed are

$$wV + wc^2 \cos a \,.\, 4a.$$

If, V being not greater than a^3, $w'a^3$ do not exceed this, the cube floats.

$\therefore$ we must have

$$\frac{w'}{w} < 1 + 4\frac{c^2}{a^2}\cos a.$$

12. Let r be the radius, h the height of the cylinder, the other symbols as in (11), wr^2 being the tension.

$$\frac{w'}{w} < 1 + 2\frac{r}{h}\cos a$$

is the condition, obtained in precisely the same way.

13. The forces on the rod are its own weight, the upward pressure of the liquid, which is equal to the weight of liquid displaced below the plane level, and the resultant downward tension which is equal to the weight of liquid elevated above the plane level.

The apparent weight is therefore greater than the real weight by the difference between the two last-mentioned forces.

When the solid depresses the liquid, the apparent weight is less than the real weight by the sum of the weights of the liquid displaced below the plane level and of the liquid required to fill up the hollow which is formed.

14. The potential energy produced by the rise of a liquid of intrinsic weight w to a height h in a tube of radius r

$$= w\pi r^2 h \times \frac{h}{2} = \frac{1}{2} w\pi r^2 h^2,$$

and from art. 169, we see that hr is constant.

15. From Example 7, we have

$$\Pi\,(R^3 - r^3 - r'^3) = 2t\,(r^2 + r'^2 - R^2).$$

But if two bubbles were to combine without change of volume, the radius of the new bubble would be greater than that of either of the original bubbles, and the surface tension would not be sufficient to maintain the equilibrium.

It follows therefore that the volume will increase, and hence from the above equation it follows that the surface will decrease.

16. If the soap-bubbles are blown from the same mixture, the tension of each is the same, and the three surfaces must therefore meet everywhere at angles of 120°.

Hence r_1, r_2 are the sides of a triangle including an angle of 60°, and r is the perpendicular from that angle on the opposite side.

Hence that opposite side $= \frac{\sqrt{3}}{2} \,.\, \frac{r_1 r_2}{r}$.

$$\therefore\ \frac{3}{4}\frac{r_1^2 r_2^2}{r^2} = r_1^2 + r_2^2 - r_1 r_2.$$

CHAPTER XI.

EXAMPLES.

1. THE resultant attraction at any point is directed to the centre of the sphere; the surfaces of equal pressure are therefore concentric spheres.

2. If a is the radius, the pressure at the distance r is $\frac{1}{2}\mu\rho\,(4a^2 - r^2)$, and the ratio is that of $4a^2 : 2a^2$.

3. Taking the figure of Ex. (1), let P be on the surface of the solid sphere, and A on the free surface of the liquid.

Then if $PA = d$, we find, by considering the equilibrium of a thin cylinder PA, that the pressure at $P = \rho f d$.

Take two parallel plane sections of the solid sphere at the distances OL, OL' from the centre, LL' being very small.

Then, if a is the radius of the solid sphere, the resultant pressure on the thin zone of surface between the planes is

$$\rho f d \,.\, 2\pi a \,.\, LL' \,.\, \frac{OL}{a}.$$

If we take two parallel sections at the finite distance NN' from each other, the resultant pressure on the zone of surface between them

$$= 2\pi\rho f d \,.\, \Sigma LL' \,.\, OL$$

$$= \pi\rho f d\,(ON'^2 - ON^2), \text{ by Leibnitz's Theorem.}$$

Also, if r, r' are the radii of the sections, the volume described

$$= \pi d\,(r'^2 - r^2) = \pi d\,(ON'^2 - ON^2).$$

MISCELLANEOUS PROBLEMS.

1. The pressure on the lower end, being equal to the weight of water displaced by the rope, is equal to the weight of one-half the immersed rope.

The tension at the middle section of the immersed rope is therefore zero, since the weight of the portion below that section is just supported by the fluid pressure on the lower end.

2. If D, E be the middle points of the sides, DE is parallel to AB, i.e. is horizontal.

$\therefore$ the line joining the centres of gravity of OAC and OBC is horizontal.

$$\begin{aligned}\therefore \text{ pressure on } OCA &: \text{pressure on } OCB \\ = \text{area of } OCA &: \text{area of } OCB \\ = \sin 2B &: \sin 2A.\end{aligned}$$

3. When the centre of gravity is in the surface, the addition of a small quantity of water will obviously raise the centre of gravity of the whole.

Further, if a small quantity A be abstracted, the C.G. of the remainder B must be above the surface, for B, together with A, which certainly has its C.G. below the surface, form a mass whose C.G. is in the surface.

Hence, since either addition or subtraction of water raises the C.G., it must be in its lowest position when in the surface of the liquid.

4. The depth of the C.G. of the curved surface is

$$h \tan a \,.\, \cos a + \tfrac{1}{3}h \sin a = \frac{4h}{3} \sin a\,;$$

$\therefore$ whole pressure on it $= \frac{4}{3}\pi w h^3 \tan^2 a$.

Whole pressure on the base $= \pi w h^2 \tan^2 a \,.\, h \sin a$;

$\therefore$ pressure on curved surface : pressure on base

$$= 4 : 3 \sin a.$$

5. Let x_r be the depth of the point at which the rth line of division meets BC.

The triangle bounded by AB and this line has (if $AB=c$) an area

$$\tfrac{1}{2}cx_r,$$

and its C.G. is at a depth $\tfrac{1}{3}x_r$.

If p be the perpendicular from C on AB,

pressure on above triangle $=\tfrac{1}{6}cx_r^2$,

pressure on whole triangle $=\tfrac{1}{6}cp^2$;

$$\therefore\ x_r^2=\frac{r}{n}p^2;$$

$\therefore$ the rth point of division of BC cuts off a fraction $\sqrt{r}/\sqrt{n}$ of the whole.

6. If ρ be the density of the solid, 2ρ, 3ρ, 4ρ are the densities of the fluids.

The mixture of equal volumes has a density 3ρ;

$\therefore$ $\tfrac{1}{3}$ of the solid is immersed in it.

The mixture of equal weights has a density

$$\frac{3}{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}\rho=\frac{36}{13}\rho;$$

$\therefore$ $\tfrac{13}{36}$ of the solid is immersed in it.

7. The volume of the cone $=\tfrac{2}{3}\pi r^3=$ that of the hemisphere;

$\therefore$ the fluids are of equal density.

8. Let h be the depth of the water, x the height of the C.G. of the water and mercury above the common surface, ρ, σ the densities.

If a small volume v is added to the mercury, the upper surface of the water is raised by the same amount as if a volume v of water were poured on the top.

If $$\rho v(h-x)-(\sigma-\rho)vx$$

be positive, the common C.G. is raised and vice versa, since a small volume of fluid of density ρ is added at a height $h-x$ above the C.G. and an equal volume at a depth x below the C.G. has its density increased by $\sigma-\rho$.

$\therefore$ If $$\rho h=\sigma x$$

no change takes place in the position of the C.G., i.e. it is at a stationary point.

Also clearly $\rho h<\sigma x$ when but little mercury is employed, hence the position is one of minimum height.

9. Let h be the whole depth of the upper fluid and let x be the length of the cylinder in it.

$$\rho x + 3\rho\,(h - x) = 2\rho h.$$
$$\therefore\ h = 2x.$$

i.e. as might have been expected, half the cylinder is in each fluid.

The pressure at a depth x in the upper fluid is proportional to ρx.

At a depth x below the common surface the pressure is proportional to

$$\rho h + 3\rho\,.\,x = 5\rho x,$$

i.e. it is five times as great as at the other end of the cylinder.

10. The centre of pressure is half-way down.

The area of the lower part cut off is therefore one quarter of the triangle.

Its C.G. is at double the depth of that of the whole triangle, and therefore the pressure on it is one-half that on the whole triangle.

11. If the point in the surface be A, the horizontal diagonal BC and the lowest point D at a depth $2z$, the depth of BC is z.

The centres of pressure of ABD and ACD are therefore each at a depth $\frac{7}{6}z$.

Now the C.G.'s of these triangles being at equal depths the pressures on them are equal and the centre of pressure of the parallelogram is at the same depth as that of the triangles, i.e. $\frac{7}{12}$ the depth of D.

12. Let x be the depth of the vertex, p the latus-rectum of the parabola.

$$\text{Area immersed} = \tfrac{4}{3}x\,.\,\sqrt{px};$$
$$\therefore\ \sigma\,.\,\tfrac{4}{3}h\,.\,\sqrt{ph} = \rho\,.\,\tfrac{4}{3}x\,.\,\sqrt{px};$$
$$\therefore\ x = h\left(\frac{\sigma}{\rho}\right)^{\frac{2}{3}}.$$

13. The original volume of air when the sphere just closes the cylinder is

$$\pi r^2\,.\,h - \tfrac{2}{3}\pi r^3.$$

If ϖ be the pressure of the air in final equilibrium, Π the atmospheric pressure, x the distance of the centre of the sphere from the base of the cylinder,

$$(\varpi - \Pi)\,\pi r^2 = W,$$

and volume of air finally $= \pi r^2\,.\,x - \frac{2}{3}\pi r^3$;

$$\therefore\ \frac{\pi r^2\,.\,x - \frac{2}{3}\pi r^3}{\pi r^2\,.\,h - \frac{2}{3}\pi r^3} = \frac{\Pi}{\varpi};$$

$$\therefore\ x=\tfrac{3}{8}r+\frac{\Pi}{\varpi}(h-\tfrac{3}{8}r),$$

$$\frac{\varpi}{\Pi}=1+\frac{W}{\pi r^2 . \Pi};$$

$$\therefore\ x=\tfrac{3}{8}r+\frac{\pi r^2 . \Pi(h-\tfrac{3}{8}r)}{W+\pi r^2 . \Pi}$$

$$=\frac{\tfrac{3}{8}rW+h\pi r^2\Pi}{W+\pi r^2 . \Pi}.$$

14. Let a be the semi-vertical angle of the cone, x the depth of axis immersed.

Volume immersed $=\frac{1}{3}x . \pi . x^2\tan^2 a$, and is constant.

Surface immersed $=\pi x^2\tan^2 a . \operatorname{cosec} a$.

Hence this latter is least when $x\sin a$ is greatest.

Now $x^3\tan^2 a$ is constant;

$\therefore$ $x\sin a$ is greatest when $\sin^3 a\cot^2 a$ is greatest,

i.e. $\sin a\cos^2 a$ is greatest.

$\sin a+\sin 3a$ is greatest when,

θ being a small change in a,

$$\sin(a+\theta)-\sin a=+\sin 3a-\sin 3(a+\theta),$$

$$\theta\cos a=-3\theta\cos 3a;$$

$$\therefore\ \cos a=-12\cos^3 a+9\cos a;$$

$$\therefore\ \cos^2 a=\tfrac{2}{3}\text{ and }\tan^2 a=\tfrac{1}{2}.$$

15. If the plane be inclined at an angle θ to the horizontal, the depth of the C.G. of the lower half below the centre is $\frac{a}{2}\cos\theta$.

$\therefore$ the pressure on lower portion : pressure on upper portion

$$=a+\frac{a}{2}\cos\theta : a-\frac{a}{2}\cos\theta$$

$$=2+\cos\theta : 2-\cos\theta.$$

If this ratio be $m:1$,

$$2m-m\cos\theta=2+\cos\theta;$$

$$\therefore\ \cos\theta=2\frac{m-1}{m+1}.$$

The greatest value of m is 3 when $\cos\theta=1$, $\therefore\ \theta=0$.

The least value of m is 1 when $\cos\theta=0$, $\therefore\ \theta=\frac{\pi}{2}$.

16. Let $4a$ be the latus-rectum, x the distance of common surface from focus. The heights of the free surface above the focus are $r-2a$, $r'-2a$, and of the common surface $x-2a$;

$$\therefore\ \rho(r-x)=\rho'(r'-x).$$

17. Let e be the thickness of each stratum.

Taking the unit of force as the weight of a pound at the place, the pressure at the lowest point is

$$\rho e(1+2+3+\ldots+n)=\frac{n(n+1)}{2}\rho e.$$

At the bottom of the rth stratum, the pressure is $\frac{r(r+1)}{2}\rho e$, and the depth $h=re$;

$\therefore$ if $\rho=\mu e$, the density will, when e is indefinitely diminished, vary as the depth, and the pressure $=\frac{r(r+1)}{2}\mu\frac{h^2}{r^2}=\frac{1}{2}\mu h^2$, when r is indefinitely increased.

18. Let x, y be the heights of the surfaces of the two fluids above the base.

Then $\qquad x+y=$ height of triangle,

and $\qquad x:y=\rho':\rho$;

$$\therefore\ x-y:x+y=\rho'-\rho:\rho'+\rho,$$

or, difference of heights of surfaces : height of triangle

$$=\rho'-\rho:\rho'+\rho.$$

19. The whole pressures are proportional to the pressures at the C.G.'s of the strata, i.e. on the rth piece the pressure is proportional to (see 16)

$$\left(\frac{r(r-1)}{2}+\frac{r(r+1)}{2}\right)\rho a,\text{ i.e. to } r^2.$$

20. The whole pressure on the curved surface is (r being the radius)

$$\pi r(1^2+2^2+3^2+\ldots n^2)\rho a^2 \text{ lbs. wt.}$$

That on the base is

$$\pi r^2\cdot\rho[1+2+3+\ldots n]a.$$

$$\therefore\ \text{whole pressure}=\pi\rho ra\left[\frac{2n+1}{3}a+r\right]\frac{n(n+1)}{2}.$$

If the fluids be mixed the density is

$$\frac{n+1}{2}\rho;$$

$\therefore$ the pressure on the curved surface is

$$\pi r \, . \, \rho \, . \, n^2 a^2 \frac{n+1}{2}.$$

$\therefore$ it is increased in the ratio

$$\frac{n^2(n+1)}{2} : \frac{n(n+1)(2n+1)}{6},$$

or $$3n : 2n+1.$$

If the densities be $\sigma \, . \, \sigma+\rho \ldots \tau+n\rho$,
the pressure at lowest point of nth stratum is

$$a[\sigma+(\sigma+\rho)+\ldots+(\sigma+n\rho)]$$
$$=na\left[\sigma+\frac{n+1}{2}\rho\right].$$

Whole pressure on curved surface

$$=\pi r \, . \, \sigma \, . \, n^2 a^2 + \pi r \, . \, \rho a^2 \frac{n(n+1)(2n+1)}{6}.$$

That on base is

$$\pi r^2 \, . \, \sigma \, . \, na + \pi r^2 \, . \, \rho a \frac{n(n+1)}{2}.$$

If $$n\rho = kh,$$
$$na = h,$$

and n be increased while ρ and a are decreased indefinitely, the pressures become

$$\pi r \, . \, \sigma h^2 + \tfrac{1}{6}\pi r k h^3 = \pi r h^2 [\sigma + \tfrac{1}{6}kh],$$

and $$\pi r^2 \sigma h + \tfrac{1}{2}\pi r^2 k h^2 = \pi r^2 h [\sigma + \tfrac{1}{2}kh].$$

21. Let x, y be the depths of the vertex below the surface in the two positions, $2a$ the vertical angle of the cone.

Then volume immersed at first $= \tfrac{1}{3}x \, . \, \pi x^2 \tan^2 a$;

$$\therefore \; y^3 = \frac{27}{8}x^3;$$

$$\therefore \; y = \tfrac{3}{2}x.$$

Volume within cone between these two positions

$$= \frac{19}{8} \, . \, \tfrac{1}{3}\pi x^3 \tan^2 a.$$

Volume of a length $y - x \left(=\frac{x}{2}\right)$ of the cylinder is

$$\pi r^2 \, . \, \frac{x}{2}.$$

Now $r^2 : x^2 \tan^2 a = 19 : 6$;

$\therefore$ volume of length $\frac{x}{2}$ of cylinder $= \frac{19}{12}\pi x^3 \tan a =$ twice volume of this portion of cone; $\therefore$ the surface of water in cylinder rises just sufficiently to keep equilibrium.

22. Let x be the distance of the C.G. of the surface from the axis, θ the angle made by the perpendicular from the C.G. on the axis with the vertical, S the area of the surface. The depression of the C.G. in turning through a right angle is

$$x(\cos\theta + \sin\theta).$$

In turning through another it is

$$x(\cos\theta - \sin\theta);$$

$$\therefore \quad wSx(\cos\theta + \sin\theta) = A,$$

$$wSx(\cos\theta - \sin\theta) = B.$$

The difference between the greatest and least pressures is

$$wS \,.\, 2x = \sqrt{2(A^2 + B^2)}.$$

23. Let $2h$ be the height, a the semi-vertical angle of the cone.

Area of surface of frustum $= \pi \,.\, 3h^2 \,.\, \tan^2 a \,.\, \operatorname{cosec} a$.

Its C.G. is at a depth x below the vertex where

$$3 \,.\, x + 1 \,.\, \tfrac{2}{3}h = 4 \,.\, \tfrac{4}{3}h;$$

$$\therefore \quad x = \tfrac{14}{9}h;$$

$\therefore$ whole pressure on curved surface

$$= w \,.\, \tfrac{14}{9}h \,.\, \pi \,.\, 3h^2 \tan^2 a \,.\, \operatorname{cosec} a$$

$$= \tfrac{14}{3}\pi w h^3 \tan a \sec a.$$

Pressure on base $= w \,.\, 2h \,.\, \pi \,.\, 4h^2 \tan^2 a$;

$$\therefore \quad \tfrac{14}{3} \operatorname{cosec} a : 8 = 7 : 6;$$

$$\therefore \quad \operatorname{cosec} a = 2;$$

$$\therefore \quad a = 30^\circ.$$

24. The weight of the contained fluid is

$$w \,.\, \frac{7}{8} \,.\, \frac{2h}{3} \,.\, \pi \,.\, 4h^2 \tan^2 a$$

$$= \tfrac{7}{3}\pi w h^3 \tan^2 a = W \text{ (say)}.$$

The upward pressure on the curved surface

$$= \tfrac{14}{3}\pi w h^3 \tan^2 a \operatorname{cosec} a \,.\, \sin a = \tfrac{14}{3}\pi w h^3 \tan^2 a.$$

The upward pressure on the cover

$$=w \,.\, \pi \,.\, h^2 \tan^2 a \,.\, h;$$

$\therefore$ whole upward pressure of fluid $= \frac{17}{3}\pi w h^3 \tan^2 a = \frac{17}{7} W$;

$\therefore$ if the weight of the vessel be less than $\frac{17}{7} W$ it will be lifted.

25. If r be the radius of the cylinder, h the height of the cone, the volume immersed is

$$\frac{1}{8} \cdot \frac{h}{3} \cdot \pi r^2 = \tfrac{1}{24} \,.\, \pi r^2 \,.\, h.$$

If the surface of the fluid in the cylinder rise a distance x, the volume of the slice x of the cylinder outside the cone + that of a slice x of the cone $= \frac{1}{8}$ vol. of cone = vol. of a slice $\frac{1}{24}h$ of the cylinder;

$$\therefore \; x = \tfrac{1}{24}h.$$

26. Let a be the depth of the fluid, h the height of the cone. Whole pressure on curved surface of cone at the place,

$$= \text{area of surface} \,.\, (a - \tfrac{1}{3}h) \,.\, \rho \text{ lbs. wt.}$$

$$\therefore \text{ vertical pressure} = \text{area of base} \,.\, \rho\,(a - \tfrac{1}{3}h),$$

$$\text{weight of cone} = \text{area of base} \,.\, \sigma \,.\, \tfrac{1}{3}h \text{ lbs. wt.}$$

Upward pressure on $B =$ vol. of $B \,.\, (\rho - \sigma)$;

$$\therefore \text{ we must have vol. of } B = \text{vol. of cone } \frac{\rho\left(\frac{3a}{h} - 1\right) + \sigma}{\rho - \sigma}.$$

27. The curve of buoyancy is a similar and similarly situated concentric ellipse, and in whatever position the ellipse is held, with its centre in the surface, the resultant fluid pressure is in the direction of the normal at the lowest point of the curve of buoyancy.

If the axis is horizontal, and the ellipse then slightly displaced, the normal at the lowest point of the buoyancy curve will intersect the minor axis above the centre of the ellipse. The equilibrium is therefore stable; and, similarly, the equilibrium is unstable when the axis is vertical.

Or we can reduce the problem to the case of an elliptic lamina, resting with its plane vertical on a horizontal plane, and then treat the question in the same manner. See Art. 69.

28. If T be tension of each string, a length of side, the moment of forces tending to keep a face in position

$$= T\sqrt{3} \times \text{height of tetrahedron} = T\sqrt{2} \,.\, a.$$

The centre of pressure on a face is halfway down.

The pressure is $w \cdot \frac{1}{3}$ height $\times \frac{a^2\sqrt{3}}{4}$ = weight of fluid.

$$\therefore\ \frac{\sqrt{3}}{4} \text{ weight of fluid} = T\sqrt{2} \cdot a.$$

$$\therefore\ T : W = \sqrt{3} : 4\sqrt{2}.$$

29. Let $w_1, w_2, \ldots$ be the weights and $\rho_1, \rho_2, \ldots$ the densities. When totally immersed they will be able to rest if the weight of the displaced liquid is equal to the sum of their weights; hence, if σ is the density of the liquid,

$$\sigma \cdot \Sigma(w/\rho) = \Sigma(w).$$

If they are required to rest in any position, we must in addition have their C. G. coincident with the C. G. of the displaced liquid, i.e. with the C. G. of homogeneous bodies of the same volume.

Hence if $x_1, x_2, \ldots$ are their distances from a fixed point in the rod, we must have

$$\frac{\Sigma(wx)}{\Sigma(w)} = \frac{\Sigma\left(\frac{w}{\rho}x\right)}{\Sigma\left(\frac{w}{\rho}\right)}.$$

Applying this condition,

$$\frac{W_1x - W_3y}{W_1 + W_2 + W_3} = \frac{\frac{W_1}{\rho_1}x - \frac{W_3}{\rho_3}y}{\frac{W_1}{\rho_1} + \frac{W_2}{\rho_2} + \frac{W_3}{\rho_3}},$$

$$x\left[W_1W_2\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) + W_1W_3\left(\frac{1}{\rho_3} - \frac{1}{\rho_1}\right)\right]$$
$$= y\left[W_1W_3\left(\frac{1}{\rho_1} - \frac{1}{\rho_3}\right) + W_2W_3\left(\frac{1}{\rho_2} - \frac{1}{\rho_3}\right)\right],$$

$$\frac{x}{W_3}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) + \frac{y}{W_1}\left(\frac{1}{\rho_3} - \frac{1}{\rho_2}\right) = \frac{x+y}{W_2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_3}\right).$$

30. It is clear that one position of equilibrium is when the rod is vertical.

If there is another let θ be the inclination of the rod to the vertical, then if ρ, ρ' are the densities of the lower and upper liquids, and σ the density of the rod,

$$\tfrac{1}{2}\rho c^2 \sec^2\theta + \rho'(a - c\sec\theta)\frac{a + c\sec\theta}{2} = \sigma\frac{a^2}{2},$$

$$\therefore\ \cos^2\theta = \frac{c^2}{a^2} \cdot \frac{\rho - \rho'}{\sigma - \rho'},$$

so that there is another position of equilibrium if

$$\sigma > \rho', \text{ and } c^2(\rho - \rho') < a^2(\sigma - \rho').$$

Considering the vertical position, give the rod a slight displacement θ;

Then the moment of the forces about the lower end varies as

$$\rho'(a^2 - c^2\sec^2\theta) + \rho c^2 \sec^2\theta - \sigma a^2,$$

or as $$c^2(\rho - \rho')\sec^2\theta - a^2(\sigma - \rho'),$$

and this is positive if $c^2(\rho - \rho') > a^2(\sigma - \rho')$,

which is the condition that the vertical position is the only one possible.

If the moment is negative, the inclined position is one of stable equilibrium.

31. Let a be the length of the cylinder, x the portion of it filled with air, Π the atmospheric pressure, h the height of the water-barometer.

The pressure of the air is $\dfrac{a}{2x}\Pi$.

$\dfrac{3a}{4} - x$ is difference of heights of water inside and outside.

$$\therefore \ \frac{a}{2x}\Pi + \frac{\frac{3a}{4} - x}{h}\Pi = \Pi.$$

$$\therefore \ 2ah + 3ax - 4x^2 = 4xh,$$

$$4x^2 + x(4h - 3a) - 2ah = 0.$$

The positive root of this equation gives the required value of x.

32. Let $3a$, $3b$ be the lengths of the sides of the parallelogram. The lines of division trisect the opposite sides of the figure.

The depths of the C.G.'s of the triangular portions are

$$\tfrac{2}{3}\cdot\left\{3\sqrt{a^2+b^2} - 2b\cdot\frac{b}{\sqrt{a^2+b^2}}\right\} = \tfrac{2}{3}\,\frac{3a^2+b^2}{\sqrt{a^2+b^2}}, \quad \text{and} \quad \tfrac{2}{3}\,\frac{3b^2+a^2}{\sqrt{a^2+b^2}}.$$

The areas are each $3ab$.

$$\therefore \ P_1 = w\,.\,3ab\,.\,\tfrac{2}{3}\,.\,\frac{3a^2+b^2}{\sqrt{a^2+b^2}} = 2wab\,.\,\frac{3a^2+b^2}{\sqrt{a^2+b^2}},$$

$$P_3 = \qquad 2wab\,.\,\frac{3b^2+a^2}{\sqrt{a^2+b^2}},$$

$$P_1 + P_2 + P_3 = w\,.\,9ab\,.\,\tfrac{3}{2}\sqrt{a^2+b^2} = \tfrac{27}{2}wab\sqrt{a^2+b^2},$$

and $$P_1 + P_3 = 8wab\,.\,\sqrt{a^2+b^2},$$

$$\therefore \ P_2 = \tfrac{11}{2}wab\sqrt{a^2+b^2},$$

$$\therefore \ P_1 : P_2 : P_3 = 4(3a^2+b^2) : 11(a^2+b^2) : 4(a^2+3b^2),$$

and $$16P_2 = 11(P_1 + P_3).$$

33. Let w be the intrinsic weight of the liquid, h the height of the cone, r the radius of the base, so that $r=h\tan a$.

The whole pressure on the curved surface is

$$w\,.\,\tfrac{2}{3}h\,.\,r^2\operatorname{cosec} a.$$

The pressure on the base is $wh\pi r^2$.

$$\therefore\ P=w\pi r^2 h\left[\frac{2+3\sin a}{3\sin a}\right],$$

$$P'=w\,.\,\tfrac{1}{3}h\,.\,\pi r^2,$$

$$\therefore\ \frac{P}{P'}=\frac{2+3\sin a}{\sin a}.$$

In the second case the depth of C.G. of surface is diminished in the ratio of 1 : $\cos\theta$.

$$\therefore\ \frac{P}{P'}=\frac{2+3\sin a}{\sin a}\,.\cos\theta,$$

P' remaining unchanged.

34. The centre of pressure of any face divides the height in the ratio 3 : 1.

If a be the length of an edge,

Height of tetrahedron $=\sqrt{\tfrac{2}{3}}\,.\,a$.

Height of a face $=\dfrac{\sqrt{3}}{2}a$.

Let W, W' be weights of the fluid and a face.

The pressure on a face

$$=w\times\tfrac{2}{3}\text{ height of tetrahedron}\times\text{area of a face}=2W.$$

$$\therefore\ W'\,.\,\frac{1}{9}\,.\,\frac{\sqrt{3}}{2}a\text{ must be not less than }2W\,.\,\frac{1}{4}\,.\,\frac{\sqrt{3}}{2}a,$$

i.e. W' not less than $\dfrac{9W}{2}$.

35. Let x be the depth of air in the inverted cone when vertex is in the surface.

Its pressure is

$$\frac{a^3}{x^3}\Pi.$$

$$\therefore\ \Pi+\frac{x}{h}\Pi=\frac{a^3}{x^3}\Pi,$$

or

$$1+\frac{x}{h}=\frac{a^3}{x^3},$$

V being the volume of the cone.

$$W=w\,.\frac{x^3}{a^3}V=\frac{\Pi}{h}\frac{x^3}{a^3}V=\frac{\Pi}{x+h}V.$$

$$(m+1)\,W=w\,V=\frac{\Pi}{h}V=\frac{x+h}{h}W.$$

$$\therefore\ m=\frac{x}{h},$$

$$\therefore\ 1+m=\frac{a^3}{m^3h^3}\text{ or }\frac{a}{h}=m\sqrt[3]{1+m}.$$

36. Let a be the length of the cylinder, x the length of it filled with air at any time, h the height of the water barometer.

If
$$\frac{a}{x}\Pi>\Pi+\frac{a-x}{h}\Pi,$$
water will flow over,
$$\text{i.e. if }\frac{a}{x}+\frac{x}{h}-\frac{a}{h}>1.$$

The limit is reached when the inequality becomes an equality,

$$\text{i.e. }x^2-(h+a)\,x+ah=0.$$

$$(x-h)(x-a)=0.$$

$$\text{i.e. }x=a\text{ or }x=h.$$

i.e. if once started, water may be poured in till its depth equals the height of the water barometer before any flows over.

If $a<h$, then $x<h$, since $x<a$ of necessity, and the piston will not begin to sink at all. The water runs over from the beginning.

37. The pressure in the interior cylinder is $\frac{a}{y}\Pi$,

$$\frac{a}{y}h+(a-y)=h+a-x.\quad\therefore\ \frac{a-y}{y}h=y-x.$$

$$x-\frac{x-y}{2}=\frac{a}{2}.$$

$$\therefore\ x+y=a,$$

$$\therefore\ \frac{y}{x}(y-x)=h.$$

$$(a-x)(a-2x)=xh,$$

$$\therefore\ 2x^2-x(3a+h)+a^2=0,$$

$$4x=3a+h\pm\sqrt{a^2+6ah+h^2},$$

$$\therefore\ 4y=a-h\mp\sqrt{a^2+6ah+h^2}.$$

38. Let C be the centre of the rectangular hyperbola.

PN a perpendicular on the transverse axis.

PTt the tangent at P to the curve, cutting the axes in T, t.

The horizontal sections of the columns of water standing on an element of surface near P in the two cases are as $Ct : CT$.

The heights of the columns are PN and CN,

$\therefore$ the pressures are as

$$PN . Ct : CN . CT = BC^2 : AC^2,$$

i.e. they are equal.

Thus since the pressures on every small portion of the area are the same in the two cases, the pressures are equal on any finite area.

39. Let a be the semi-vertical angle, $2h$ the height of either cone.

The area of the surface of the lower cone is

$$4\pi h^2 \tan a . \sec a.$$

$$\therefore \text{ the whole pressure on it is } \frac{28}{3} w\pi h^3 \sin a.$$

$$\therefore \text{ the resultant pressure is } \frac{28}{3} w\pi h^3 \tan^2 a.$$

The weight of the fluid in the upper cone is

$$\tfrac{1}{3} w\pi h^3 \tan^2 a.$$

If W be the weight of fluid either cone can contain, W' the weight of either cone,

$$W = \frac{8}{3} w\pi h^3 \tan^2 a.$$

$$2W' + \tfrac{1}{3} w\pi h^3 \tan^2 a = \frac{28}{3} w\pi h^3 \tan^2 a,$$

$$\therefore\ 2W' = \frac{27}{3} w\pi h^3 \tan^2 a = \frac{27}{8} W.$$

$$\therefore\ W' : W = 27 : 16.$$

40. Let a be the side of the square and let $CE = b$.

The whole pressure on the square $= \tfrac{1}{2} wa^3$.

The whole pressure on the triangle BCE is

$$\tfrac{1}{2} ab . \tfrac{2}{3} wa = \tfrac{1}{3} wa^2 b\,;$$

$$\therefore\ \tfrac{1}{3} b = \tfrac{1}{4} a\,; \quad \therefore\ b = \tfrac{3}{4} a.$$

$\therefore$ the distance of the centre of pressure of BCE from BC

$$= \tfrac{3}{4} . \tfrac{3}{8} a = \tfrac{9}{32} a.$$

Its distance from $CD = \frac{1}{4}a$.

The centre of pressure of the square is $\frac{1}{3}a$ from CD;

$\therefore$ the distance between these two $= a\sqrt{(\frac{1}{12})^2 + (\frac{7}{32})^2}$

$$= \frac{a}{96}\sqrt{505}.$$

Since the centre of pressure of the other part of the square and that of the triangle must be equidistant from the centre of pressure of the whole square, the distance between the two centres of pressure

$$= \frac{a}{48}\sqrt{505}.$$

41. Let a be the side of the square, θ the inclination at which the moveable side is inclined to the horizontal, and W its weight;

then, taking moments about the hinge,

$$W.\frac{a}{2}\cos\theta = wa^2.\frac{a}{2}\sin\theta.\frac{a}{3};$$
$$\therefore \tan\theta = \frac{3W}{wa^3}.$$

42. Let $\theta + \alpha$, $\theta - \alpha$ be the inclinations of the two radii to the radius in the surface, a the radius.

The C.G. of the sector is on the bisecting radius at a distance $\frac{2}{3}.\frac{a\sin\alpha}{\alpha}$ from the centre;

$\therefore$ its depth is $\frac{2}{3}\frac{a\sin\alpha\sin\theta}{\alpha}$;

$\therefore$ the pressure is $w.\frac{2}{3}\frac{a\sin\alpha\sin\theta}{\alpha}.\alpha a^2$

$$= \tfrac{2}{3}wa^3\sin\alpha\sin\theta = \tfrac{1}{3}wa^3\{\cos(\theta-\alpha) - \cos(\theta+\alpha)\}.$$

The pressure on the whole $= \frac{2}{3}wa^3$.

If $\theta = \alpha$ and the pressure on the sector is $\frac{1}{6}wa^3$,

$$\tfrac{1}{2} = 1 - \cos 2\theta = 2\sin^2\theta;$$
$$\therefore \sin\theta = \tfrac{1}{2}; \quad \therefore \theta = 30°.$$

$\therefore$ the bounding radius makes an angle of 60° with the radius in the surface.

43. Since the cylinder may be divided into narrow vertical strips, the centre of pressure of each of which divides its length in the ratio 2 : 1, the centre of pressure of the whole is at this same depth.

It is also obviously at the C.G. of the section of the cylindrical surface at this depth.

44. In each case the centre of pressure must be at the same depth as that of the triangle intercepted by the two given planes on the vertical plane through the axis perpendicular to the given radius;

$\therefore$ it divides the height in the ratio 3 : 1 in the first case.

It bisects the height in the second case.

45. Let h be the height of the pyramid, A the area of the base, d the depth of the centre of the base.

The horizontal pressure on the inclined surfaces = that on the base $= wAd$.

The vertical pressure on the inclined surfaces = weight of fluid displaced $= \frac{1}{3}wAh = W$ (say);

$$\therefore \text{ Resultant pressure} = W\sqrt{1+\frac{9d^2}{h^2}}.$$

And it is inclined at an angle $\tan^{-1}\frac{3d}{h}$ to the vertical.

If the base be inclined to the vertical at an angle θ,

The horizontal component of the pressure

$$= wAd\cos\theta = \frac{3Wd}{h}\cos\theta.$$

The vertical component $= W \pm \frac{3Wd}{h}\sin\theta$.

$\therefore$ the resultant $= W\sqrt{1\pm\frac{6d}{h}\sin\theta+\frac{9d^2}{h^2}}$.

And it is inclined to the vertical at an angle

$$\tan^{-1}\frac{3d\cos\theta}{h\pm 3d\sin\theta}.$$

The upper sign applies if the vertex is depressed below the centre of the base, the lower in the contrary case.

46. If h be the perpendicular distance of the base from the vertex, a the radius of the base, d the depth of the centre of the base,

Horizontal pressure on curved surface $= w\pi a^2 . d$.

Vertical pressure $= \frac{1}{3}w\pi a^2 . h = W$, the weight of fluid displaced;

$$\therefore \text{ Resultant pressure} = W\sqrt{1+\frac{9d^2}{h^2}}.$$

47. The resultant pressure on the fluid is equal to its weight and acts vertically through its centre of gravity, and this is equal and opposite to the resultant pressure on the curved surfaces of the cone.

Now the C.G. of the fluid is in a line joining the vertex to the centre of the base and divides that line in the ratio 1 : 3.

$\therefore$ if the vertical through this point be drawn downwards to meet the curved surface, it will meet it at the required centre of pressure, while the pressure is equal to the weight of the contained fluid.

48. Since the diameter through the point of contact bisects all horizontal chords, the centre of pressure always lies in that diameter.

Let θ be the angle between that diameter and the horizontal.

If QVQ' be any ordinate, PV being the diameter,

$$QV^2 = 4SP \,.\, PV = 4AS \operatorname{cosec}^2\theta \,.\, PV$$
$$= 4AS \,.\, \operatorname{cosec}^3\theta \,.\, (\text{depth of } V).$$

$\therefore$ if the area be divided by a number of horizontal lines which remain always at the same depths, the portions between any two such consecutive lines are increased in area in the ratio $\operatorname{cosec}^{\frac{3}{2}}\theta : 1$.

$\therefore$ all the pressures being increased in the same proportion, the centre of pressure remains at the same depth.

49. The resultant horizontal pressure on the part described is equal to that on the corresponding portion of the triangular plane face of the half cone.

Now the depth of the C.G. of a cone and of the centre of pressure of an isosceles triangle whose vertex is in the surface, is in each case $\frac{3}{4}$ of the height, and they are therefore in this case in the same horizontal line.

From this the required result is obvious, on drawing a figure.

50. Let $2d$ be the depth, A the area of the cylinder, $2W$ the weight of water it would contain.

Its weight $= W = wAd$.

Let x be the depth of the stop.

The pressure of the air inside exceeds that outside by

$$W/A = wd.$$

If h be the height of the water barometer, y the depth of the water below the stop,

$$\frac{h+x+y}{h} = \frac{d}{y} = \frac{h+d}{h};$$

$$\therefore\ x = d - \frac{hd}{h+d}.$$

If there be a hole in the stop and u be now its depth, v the length of the column of air in the cylinder,

$$\frac{h+u-d+v}{h}=\frac{2d}{v}=\frac{h+d}{h};$$

$$u=2d-\frac{2hd}{h+d}=2x. \qquad \text{Q. E. D.}$$

51. At temperature zero let the portion not immersed occupy a length z of the tube.

Then $$z\left(1+\frac{\tau}{6480}\right)=m.$$

If t' be the real temperature of the liquid, we should have (on wholly immersing the instrument),

$$t'=t-m+z\left(1+\frac{t'}{6480}\right);$$

$$\therefore\ t'-t=-m+\left(1+\frac{t'}{6480}\right)\frac{m}{1+\frac{\tau}{6480}}$$

$$=m\left\{\frac{t'+6480}{\tau+6480}-1\right\}=m\frac{t'-\tau}{\tau+6480}$$

$$=m\frac{t'-t}{\tau+6480}+m\frac{t-\tau}{\tau+6480},$$

$$\therefore\ t'-t=\frac{m(t-\tau)}{6480+\tau-m}.$$

52. Let h be the height, $2a$ the vertical angle of the cone.

Horizontal pressure on the base $=w\pi h^3\tan^3 a$.

Vertical pressure on the curved surface $=\frac{1}{3}w\pi h^3\tan^2 a=W$.

$$\therefore\ \text{the resultant}=W\sqrt{1+9\tan^2 a},$$

and makes an angle $\tan^{-1}(3\tan a)$ with the vertical.

It acts through the C.G. of the cone;

$\therefore$ (1) If its direction passes through C

$$\frac{h}{3}\bigg/h\tan a=3\tan a;$$

$$\therefore\ \tan a=\tfrac{1}{3}.$$

(2) If it is parallel to a generator

$$\cot a=3\tan a;$$

$$\therefore\ \tan a=\frac{1}{\sqrt{3}},\quad \therefore\ a=30^\circ.$$

It can never be perpendicular to a generating line, for in that case it would make an angle α with the vertical. This could only be if $\alpha=0$, i.e. the cone would become a straight line.

53. Let b be the distance of the centre of the hollow from the centre of the sphere, c its radius, w_1, w_2 the intrinsic weights of the sphere and liquid.

To remove the hollow from the highest to the lowest position is equivalent to raising a volume $\frac{4}{3}\pi c^3$ of the material of the sphere a height $2b$ and lowering a volume $\frac{4}{3}\pi c^3$ through the same distance.

$\therefore$ If $2w_1 > w_2$

energy is required to be supplied for this purpose, i.e. the C.G. of the whole is raised and the position of stable equilibrium is with the hollow uppermost.

54. Let h be the height of the water barometer, s the specific gravity of mercury, a the depth of the top of the bell, b its length, x the part containing air.

$$\frac{h+a+x}{h}=\frac{b}{x} \quad \text{(i)}.$$

The height of the barometric column is $(h+a+x)/s$ in the bell, where x is determined from equation (i).

(i) If the wood come from outside, the level of the water is slightly depressed, and hence the pressure and the height of the barometer slightly increased.

(ii) The water slightly rises, the pressure and the height of the barometer slightly decreasing.

55. The areas of the two portions of the surface are as 3 : 1.

Let ρ, ρ' be the densities of the fluids.

The pressures at the depths of the C.G.'s of the two portions are as

$$\tfrac{4}{3}\rho : \rho+\tfrac{1}{3}\rho';$$

$\therefore$ the whole pressures are as

$$\tfrac{4}{3}\rho : \rho+\tfrac{1}{3}\rho'=4\rho : 3\rho+\rho'.$$

56. Let $x : y$ be the ratio in which the sides are divided. Then the pressures being equal at the two ends of the horizontal side,

$$x+3y=y+2x;$$

$$\therefore 2y=x,$$

or

$$x : y=2 : 1.$$

57. If A is the vertex of the triangle, D the middle point of the base BC, and E the middle point of the line of flotation, we have to express the fact that DE is vertical.

If a is the side of the triangle, and if x, y are the lengths of the sides immersed, the condition is

$$\frac{\frac{a}{2}-\frac{y}{2}}{\frac{a}{2}-\frac{x}{2}}=\frac{x}{y}, \text{ or } ax-x^2=ay-y^2,$$

so that either $$x=y, \text{ or } x+y=a,$$
and the other equation is $$\tfrac{1}{2}\rho xy=\tfrac{1}{2}\sigma a^2.$$

58. The pressure on the base remains constant, R (say).

If W be the weight of water displaced

$$(R\sqrt{1-s^2})^2+(W+Rs)^2=P^2;$$
$$\therefore\ W^2+R^2+2WRs=P^2.$$

Similarly $$W^2+R^2+2WRs'=P'^2,$$
$$W^2+R^2+2WRs''=P''^2;$$
$$\therefore\ P^2(s'-s'')+P'^2(s''-s)+P''^2(s-s')=0.$$

59. If θ is the inclination of the axis to the vertical

$$\tan\theta=h\tan a\div\frac{h}{4}=4\tan a.$$

The pressure on the base $=w\pi h^3\tan^3 a\sin\theta$.

That on the curved surface

$$=w\pi h^2\tan^2 a\operatorname{cosec} a\left(h\tan a\sin\theta+\frac{h}{3}\cos\theta\right).$$

The ratio of these is

$$\begin{aligned}&\tan a\sin\theta : \operatorname{cosec} a(\tan a\sin\theta+\tfrac{1}{3}\cos\theta)\\ &=4\tan^2 a : \operatorname{cosec} a\{4\tan^2 a+\tfrac{1}{3}\}\\ &=12\sin^3 a : 12\sin^2 a+\cos^2 a\\ &=12\sin^3 a : 1+11\sin^2 a.\end{aligned}$$

60. If θ is the inclination of the axis to the vertical the horizontal pressure on the curved surface

$$=w\pi h^3\tan^3 a\sin^2\theta.$$

The vertical pressure is

$$\tfrac{1}{3}w\pi h^3\tan^2 a+w\pi h^3\tan^3 a\sin\theta\cos\theta;$$

$$\therefore\ \tan\phi=\frac{\tan a\sin^2\theta}{\frac{1}{3}+\tan a\sin\theta\cos\theta}$$

$$=\frac{16\tan^3 a}{\frac{1}{3}(1+16\tan^2 a)+4\tan^2 a}=\frac{48\tan^3 a}{1+28\tan^2 a},$$

or $$\cot\phi=\frac{28\cot a+\cot^3 a}{48}.$$

61. The only effect of the fluid is to reduce the apparent weight of the chain, for the resultant fluid pressure on each link is vertical. Thus the form remains the same as in air.

62. Let the surface divide the generating line in the ratio $x : 1-x$, and let θ be the angle which the axis makes with the vertical, $2a$ the vertical angle of the cone;

$$\therefore\ \cos 2a = \tfrac{2}{3}.$$

If l be the length of the side of the cone, the centre of the liquid surface is distant horizontally from the vertex

$$\tfrac{1}{2}\{xl \sin(\theta - a) + l \sin(\theta + a)\};$$

$\therefore$ the C.G. of the fluid is distant horizontally from the vertex

$$\frac{3l}{8}\{x \sin(\theta - a) + \sin(\theta + a)\}.$$

The point of suspension being vertically over this,

$$l \sin(\theta - a) = \frac{3l}{8}\{x \sin(\theta - a) + \sin(\theta + a)\}.$$

But
$$x = \frac{\cos(\theta + a)}{\cos(\theta - a)} = \frac{\sin 2\theta - \sin 2a}{\sin 2(\theta - a)};$$

$$\therefore\ 8 \sin(\theta - a)\cos(\theta - a) = 3\{\sin(\theta - a)\cos(\theta + a) + \sin(\theta + a)\cos(\theta - a)\},$$

$$4 \sin 2(\theta - a) = 3 \sin 2\theta;$$

$$\therefore\ \tan 2\theta = \frac{4 \sin 2a}{4 \cos 2a - 3} = -4\sqrt{5},$$

$$\sin 2\theta = \frac{4\sqrt{5}}{\sqrt{1+80}} = \frac{4\sqrt{5}}{9};$$

$$\therefore\ \sin 2(\theta - a) = \frac{\sqrt{5}}{3} = \sin 2a;$$

$$\therefore\ x = \tfrac{4}{3} - 1 = \tfrac{1}{3};$$

$$\therefore\ x : 1 - x = 1 : 2.$$

63. Let ρ, σ be the densities, θ the inclination of the major axis to the vertical.

Then the pressures at the common surface being equal in the two fluids,

$$\rho(b \sin\theta + a \cos\theta) = \sigma(a \cos\theta - b \sin\theta);$$

$$\therefore\ \tan\theta = \frac{\sigma - \rho}{\sigma + \rho} \cdot \frac{a}{b}.$$

64. Let h be the depth of the C.G. of the base.

The pressure on it is whA.

If θ be its inclination to the horizon, the pressure on the curved surface has for its

horizontal component $whA \sin\theta$,

vertical component $whA \cos\theta + wV$.

$\therefore$ the resultant pressure is $w[V^2 + 2hAV\cos\theta + h^2A^2]^{\frac{1}{2}}$,

$$P_1^2 = w^2[V^2 + 2xAV\cos\theta + x^2A^2],$$

$$P_2^2 = w^2[V^2 + 2yAV\cos\theta + y^2A^2],$$

$$P_3^2 = w^2[V^2 + 2zAV\cos\theta + z^2A^2];$$

$$\therefore\ P_1^2(y-z) + P_2^2(z-x) + P_3^2(x-y)$$

$$= w^2A^2[x^2(y-z) + y^2(z-x) + z^2(x-y)]$$

$$= w^2A^2(z-y)(y-x)(x-z).$$

65. Consider the equilibrium of a small element s just beneath the surface and of a small element s' just above the surface.

Let w, w' be the intrinsic weights of the liquid and the chain.

Let t be the tension at the surface and θ, θ' the small angles between the tangents at the two ends of the elements.

Then resolving, for each element, along the normal at the other end, we obtain, if a is the inclination to the horizontal of the string at the surface,

$$t\sin\theta = w'ks\cos(a+\theta),$$

$$t\sin\theta' = (w'-w)ks'\cos(a-\theta').$$

But if r, r' are the radii of curvature, $s = r\theta$, $s' = r'\theta'$; $\therefore$ making θ and θ' indefinitely small,

$$\frac{1}{r} : \frac{1}{r'} = w' : w' - w = \rho : \rho - \sigma.$$

66. See Chap. IV. Ex. 25.

67. If ϖ_1 ϖ_2 be the new pressures, τ the increase of temperature,

$$\frac{\varpi_1}{273+t+\tau} = \frac{\Pi}{273+t},$$

$$\frac{\varpi_2}{273+t'+\tau} = \frac{\Pi}{273+t'};$$

$$\therefore\ \varpi_1 = \Pi\left(1 + \frac{\tau}{273+t}\right),$$

$$\varpi_2 = \Pi\left(1 + \frac{\tau}{273+t'}\right).$$

$\therefore$ the increase of pressure is greater in that which had the lower temperature originally.

The pressure at zero in one will be

$$\frac{273\,\Pi}{273+t}=\Pi\left(1-\frac{t}{273}\right) \text{ approximately.}$$

That in the other will be $\Pi\left(1-\dfrac{t'}{273}\right)$.

When the air in both is forced into the same vessel, the pressure will be the sum of these two, i.e. $\Pi\left(2-\dfrac{t+t'}{273}\right)$.

68. Let Π, ϖ be the two pressures and $a=\frac{1}{273}$,

$$\frac{\Pi}{273}=\frac{\varpi n^3}{273+t};$$

$$\therefore\ \Pi : \varpi = n^3 : 1+at.$$

69. The mercury expands till it fills

$$\tfrac{1}{2}(1\cdot0036)=\cdot5018 \text{ of the vessel.}$$

$\therefore$ the air now fills $\cdot4982$ of the vessel.

If Π, ϖ be the pressures of the air at 0° and 20°,

$$\frac{\Pi\times\cdot5}{273}=\frac{\varpi\times\cdot4982}{293};$$

$$\therefore\ \frac{\varpi}{\Pi}=\frac{293}{273\times\cdot9964}=1\cdot07716\ldots$$

70. Let V be the volume of a given mass of mercury at 68°.

At 212° its volume is $\frac{70}{69}V$.

The volume of an equal mass of water at 68° is $13\cdot568\,V$.

At 212° it is $13\cdot704\times\frac{70}{69}V$.

$\therefore$ the proportional expansion of water is

$$\frac{13\cdot704\times70-13\cdot568\times69}{13\cdot568\times69}=\frac{23088}{936192},$$

$$=\cdot02466\ldots=\tfrac{2}{81} \text{ nearly.}$$

71. The area of the surface as far as the rth plane

$$=\frac{h_r}{a}\times \text{whole surface.}$$

$\therefore$ the whole pressure on this part $=\dfrac{h_r^2}{a^2}\times$ pressure on whole.

$$\therefore\ \frac{h_r^2}{a^2}=\frac{r}{n},\quad\text{or}\quad \frac{h_r}{a}=\sqrt{\frac{r}{n}}.$$

72. If the hexagon be divided into six triangles by joining the angular points to the centre, and if a be the depth of the centre, the depths of the centres of gravity are

$$\frac{a}{3}, \frac{2a}{3}, \frac{4a}{3}, \frac{5a}{3}.$$

$\therefore$ the whole pressures are in these ratios.

The centres of pressure are at depths

$$\frac{a}{2}, \frac{3a}{4}, \frac{11a}{8}, \frac{17a}{8}. \quad \text{(See p. 207.)}$$

The depth of the centre of pressure of the whole hexagon is

$$a \cdot \frac{\frac{1}{2}+2 \cdot 2 \cdot \frac{3}{4}+2 \cdot 4 \cdot \frac{11}{8}+5 \cdot \frac{17}{10}}{1+2 \cdot 2+2 \cdot 4+5}=\frac{23}{18}a.$$

73. The triangle may be divided into a number of small triangles with equal bases and the same vertex.

The centre of pressure of each of these is $\frac{3}{4}$ its length from the vertex.

$\therefore$ the centre of pressure of the whole is in the generator of the cylinder which passes through the vertex of the triangle, and divides that generator in the ratio 3 : 1.

74. Let 2θ be the angle between the planes, $2a$ the vertical angle of the cone, and h its height.

The pressure on each plane face is $\frac{1}{6}wh^3 \tan a$.

$\therefore$ the resultant horizontal pressure on the curved surface

$$=\frac{1}{3}wh^3 \tan a \sin \theta.$$

The resultant vertical pressure $=\frac{1}{3}w\theta h^3 \tan^2 a$.

$\therefore$ the resultant pressure $=\frac{1}{3}wh^3 \tan a \sqrt{\sin^2 \theta+\theta^2 \tan^2 a}$.

The c. g. of the surface is at a distance from the centre

$$\frac{2}{3} \cdot \frac{h \tan a \sin \theta}{\theta}.$$

$\therefore$ the c.g. of the contained fluid (through which the resultant pressure acts) is distant $\frac{1}{2}\frac{h \tan a \sin \theta}{\theta}$ from the axis, and $\therefore$ the line joining it to the centre of the base of the cone makes an angle $\tan^{-1}\frac{2 \tan a \sin \theta}{\theta}$ with the vertical.

The resultant pressure makes an angle

$$\tan^{-1}\frac{\sin \theta}{\theta \tan a} \text{ with the vertical.}$$

$\therefore$ its line of action passes through the centre of the base if

$$\tan^2 a=\tfrac{1}{2}, \text{ i.e. if } a=45°.$$

75. Considering the triangle of forces for each sphere it follows that the vertical through the point, O, bisects the distance between the spheres.

If u is the distance between O and the middle point,

$$2u^2+\frac{x^2}{2}=r^2+r'^2.$$

If θ is the inclination to the horizontal of the line joining the spheres,

$$r^2=u^2+\frac{x^2}{4}\mp ux\sin\theta, \qquad r'^2=u^2+\frac{x^2}{4}\pm ux\sin\theta\,;$$

$$\therefore\ \sin\theta=\frac{r^2\sim r'^2}{2ux}=\frac{r^2\sim r'^2}{x\sqrt{2(r^2+r'^2)-x^2}}.$$

Also, if P is the excess of fluid pressure over the weight of a sphere

$$\phi(x) : P :: \frac{x}{2} : u.$$

76. Let A be the area of the piston, Π the atmospheric pressure, x the length of the string in equilibrium.

The pressure on the piston is $\frac{a}{x}\Pi\,.\,A$.

$$\therefore\ x=l\left[1+\frac{\frac{a-x}{x}\Pi A}{\Pi A}\right]=\frac{la}{x}\,;$$

$$\therefore\ x=\sqrt{la}.$$

77. If z be the depth of the centre of pressure

$$2z(a+b+c)=a^2+b^2+c^2+bc+ca+ab.$$

The depth of the centre of gravity is $\frac{a+b+c}{3}$.

$$\therefore\ z-\frac{a+b+c}{3}=\frac{a^2+b^2+c^2+bc+ca+ab}{2(a+b+c)}-\frac{a+b+c}{3}$$

$$=\frac{a^2+b^2+c^2-bc-ca-ab}{6(a+b+c)}=\frac{(b-c)^2+(c-a)^2+(a-b)^2}{12(a+b+c)}.$$

78. If one point of the disc of radius R were in the surface, the centre of pressure would be at a distance $\frac{p}{r}R$ from the centre.

The pressure would be $w\,.\,\pi R^3$.

If liquid be now added till the surface is raised to a distance h above the centre, the pressure is increased to $w\pi R^2\,.\,h$, the moment about the horizontal line through the centre being unaltered, since the resultant of the additional pressures acts through the centre.

If q be now the distance of the centre of pressure from the centre,

$$w\pi R^2 \,.\, h \,.\, q = w\pi R^3 \,.\, \frac{p}{r} R,$$

or

$$q = pR^2 \div hr.$$

79. If the mercury rise through a distance x, the air which originally filled a space $A+\kappa l$ expands so as to fill a space

$$A+\kappa(l-x)+B.$$

Its pressure becomes $\frac{h-x}{h}$ of its original value.

$$\therefore \; h(A+\kappa l) = (h-x)\{A+B+\kappa l-\kappa x\}.$$

If we neglect κx^2 we have

$$x = \frac{hB}{A+B+\kappa(l+h)} = \frac{hB}{A+B}\left[1-\frac{\kappa(l+h)}{A+B}\right] \text{ q. p.}$$

Writing this value for x in the neglected term and inserting it,

$$x = \frac{hB}{A+B}\left[1-\kappa\frac{l+h}{A+B}\right] + \kappa\frac{h^2B^2}{(A+B)^3} = \frac{hB}{A+B}\left[1-\kappa\frac{Ah+(A+B)l}{(A+B)^2}\right].$$

80. Let θ be the angle made by the base of the hemisphere with the vertical, when the attached weight on the rim is just in the surface of the water.

The volume of water displaced is $\frac{(1-\cos\theta)^2(2+\cos\theta)}{2} \times$ volume of hemisphere.

If w, W be the weights of the hemisphere and of the water which would fill it

$$\tfrac{5}{4}w = \tfrac{1}{2} \,.\, W(1-\cos\theta)^2(2+\cos\theta).$$

Also taking moments about the centre (through which the fluid pressures pass)

$$\frac{w}{4} \,.\, a\sin\theta = w \,.\, \frac{a}{2}\cos\theta;$$

$$\therefore \; \tan\theta = 2,$$

$$\therefore \; \cos\theta = \frac{1}{\sqrt{5}},$$

$$\therefore \; (1-\cos\theta)^2(2+\cos\theta) = \frac{10\sqrt{5}-14}{5\sqrt{5}},$$

$$\therefore \; W : w = 25\sqrt{5} : 20\sqrt{5}-28.$$

81. Let the external atmospheric pressure increase till the water-barometer reading is $H+u$.

Let d be the depth of the top of the bell originally.

The internal pressure is $\frac{h}{x}H$ and also $H+d+x$.

$$\therefore \quad hH=x(H+d+x).$$

If the bell be free to move, it will continue to displace the same volume of water, i.e. x remains unaltered.

Let y be the amount of its motion,

$$h(H+u)=x[H+u+d+y+x]=hH+x(u+y),$$

$$y=u.\frac{h-x}{x}.$$

If the bell be held fixed, let z be the fall of the water in it. Then

$$h(H+u)=(x+z)[H+u+d+x+z]$$
$$=hH+x(u+z)+z[H+d+x+u-z].$$

Neglecting squares of small quantities

$$z=u.\frac{h-x}{H+d+2x};$$

$$\therefore \quad y:z=H+d+2x:x=x+\frac{hH}{x}:x=Hh+x^2:x^2.$$

82. If k is the depth immersed and if w, w' are the intrinsic weights of the liquid and pyramid,

$$w'h^3=wk^3.$$

The horizontal pressure of the liquid perpendicular to the dividing plane is $wk.\frac{k}{h}a.\frac{k}{3}$, and the centre of pressure is at the distance $\frac{k}{2}$ from the hinge.

$\therefore$ the moment of the horizontal pressure about the hinge

$$=\tfrac{1}{6}w\frac{ak^4}{h}=\tfrac{1}{6}w'ah^2k.$$

The vertical pressure on the half-pyramid

$$=\text{weight of half-pyramid}=\tfrac{2}{3}w'a^2h.$$

The centre of gravity of the half-pyramid is $\frac{3}{8}a$ from the dividing plane.

The line of action of the vertical pressure is $\frac{3}{8}\frac{k}{h}a$ from the plane.

$\therefore$ in order that the parts may remain in contact,

$$\frac{1}{6}ah^2k+\frac{2}{3}a^2h.\frac{3}{8}\frac{k}{h}a>\frac{2}{3}a^2h.\frac{3}{8}a,$$

or

$$k(2h^2+3a^2)>3a^2h,$$

or

$$\frac{w'}{w}>\left(\frac{3a^2}{2h^2+3a^2}\right)^3.$$

83. Let $ABCDE$ be the pentagon, A being the lowest vertex, and AF the perpendicular from A upon CD.

Then, if c is the side of the hexagon,

$$\frac{a}{2}=AF=c\cos 54^\circ+c\cos 18^\circ=2c\cos 36^\circ.\cos 18^\circ.$$

The depth of A below the surface being a, the

$$\text{depth of } C \text{ and } D \text{ is } \frac{a}{2}, \text{ and}$$

$$\text{depth of } B=\frac{a}{2}+c\sin 72^\circ=a\,\frac{\sqrt{5}+1}{4}.$$

Now it is shewn on page 177 that, if α, β, γ are the depths of the angular points of a triangle, and z the depth of its centre of pressure,

$$2z(\alpha+\beta+\gamma)=\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta.$$

Employing this formula we find that the depths of the centres of pressure of ACD and ABC are respectively

$$\frac{11}{16}a \text{ and } \frac{5+\sqrt{5}}{7+\sqrt{5}}a.$$

Let α represent the area ABC;

then area $ACD=\alpha.\dfrac{AF.CF}{\frac{1}{2}c^2\sin 108^\circ}=\alpha\,\dfrac{a^2}{2c^2}\,\dfrac{\tan 18^\circ}{\sin 72^\circ}=\alpha\,\dfrac{\sqrt{5}+1}{2}.$

The depths of the centroids of ACD and ABC are respectively $\dfrac{2}{3}a$, and $\dfrac{a}{12}(7+\sqrt{5})$.

Hence, if $\bar{z}$ is the depth of the centre of pressure of the pentagon,

$$\bar{z}\left\{\frac{\sqrt{5}+1}{3}\,\alpha a+\frac{7+\sqrt{5}}{6}\,\alpha a\right\}=\frac{\sqrt{5}+1}{3}\,\alpha a.\frac{11a}{16}+\frac{7+\sqrt{5}}{6}\,\alpha a.\frac{5+\sqrt{5}}{7+\sqrt{5}}\,a,$$

and $\therefore$

$$\bar{z}=\frac{a}{48}(29+3\sqrt{5}).$$

84. Let h be breadth of quadrilateral.

Depths of centres of pressure of ACD and ABC are $\frac{3}{4}h$ and $\frac{1}{2}h$.

Pressures on ACD and ABC are

$$\tfrac{1}{3}wh^2.CD \text{ and } \tfrac{1}{6}wh^2.AB.$$

$\therefore$ depth of centre of pressure

$$=\frac{\frac{3}{4}.\frac{1}{3}CD+\frac{1}{2}.\frac{1}{6}AB}{\frac{1}{3}CD+\frac{1}{6}AB}\,h=\frac{3CD+AB}{4CD+2AB}\,h.$$

$\therefore$ it divides the breadth in the ratio

$$3CD+AB : CD+AB,$$

and will be at the intersection of AC and BD if

$$\frac{3CD+AB}{CD+AB}=\frac{AB}{CD} \text{ or } AB^2=3CD^2.$$

85. Let v be the volume of water displaced in the position of equilibrium, and $v+ky$ the volume of the other liquid displaced when there is equilibrium;

then $$w=62{\cdot}5v=62{\cdot}5s\,(v+ky).$$

Placing the hydrometer in the liquid so that v is displaced, and letting it go, the acceleration when it has descended through the space x

$$=\frac{w-62{\cdot}5s\,(v+kx)}{m}\propto y-x.$$

The motion is therefore the same as that of a particle attracted to a centre of force, the force of attraction being proportional to the distance.

Hence the hydrometer will descend to the distance $2y$,

$$\text{i.e. } \frac{2\,(1-s)}{ks}\cdot\frac{w}{62{\cdot}5}.$$

If we take the unit of weight to be the weight of a cubic foot of water, we obtain the stated result.

86. Let r be the radius of the sphere, x the depth of its centre below the surface of the water.

The distance between the centre and the plane of contact, being the sub-normal, is $2a$.

Also $$r^2=4a\,(a+c).$$

The area of the circle of contact is $4\pi ac$.

The pressure on the sphere is equal to the weight of the water which would be contained in a cylinder whose base is the circle of contact, and whose height is $x+2a$, together with that contained by the segment of the sphere cut off by the plane of contact. The volume of this segment is

$$\tfrac{2}{3}\pi r^2\,(r-2a)-\tfrac{1}{3}\,.\,2a\,.\,4\pi ac,$$

$$\therefore\ \tfrac{2}{3}\pi r^3-\tfrac{4}{3}\pi r^2a-\tfrac{8}{3}\pi a^2c+4\pi ac\,(x+2a)=\tfrac{2}{3}\pi r^3.$$

Using the above value of r^2, this gives $x=4a^2/3c$.

87. The volume of air in the room is 4200 cubic feet.

$\therefore$ when the barometer falls from 30 to 29 inches $\frac{1}{30}$ of this leaves the room.

The weight is therefore that of

$$\frac{4200\times 1728}{30}\times\frac{33}{100}\text{ grains}=11\text{ lbs. }2833\frac{3}{5}\text{ grains}=11{\cdot}4048\text{ lbs.}$$

88. Let $2\theta_r$ be the angle made by the rth bounding radius with the surface, and let there be n sectors, a being the radius.

The area of the first r sectors is $a^2\theta_r$.

The depth of the centre of gravity is $a\dfrac{\sin^2\theta_r}{\theta_r}$.

$$\therefore\ w\,.\,a^3\sin^2\theta_r=\frac{r}{n}\,w\,.\,\frac{\pi a^2}{2}\,.\,\frac{2a}{\pi},$$

$$\sin^2\theta_r=\frac{r}{n},$$

or the rth radius makes an angle $2\sin^{-1}\sqrt{\dfrac{r}{n}}$ with the surface.

89. Let h be the height of the cylinder, a its sectional area.

The pressure on the conical surface when it is uppermost is

$$w\,.\,\tfrac{2}{3}a\,.\,3h=2wah.$$

When it is lowest it is

$$wah+\tfrac{1}{3}wa\,.\,3h=2wah.$$

90. The weight of the air in the balloon is that of $\dfrac{33}{100}$ grains, (see Ex. 87), and the weight of a cubic inch of water is therefore that of 264 grains.

At a depth x the balloon displaces $\dfrac{33}{33+x}$ cubic inches of water, the weight of which is that of $\dfrac{33}{33+x}\times 264$ grains.

The weight of the lead in water is

$$\left(1-\frac{80}{912}\right)100=\frac{5200}{57}\text{ grains wt.}$$

$$\therefore\ \frac{5200}{57}+\frac{25}{72}=\frac{33}{33+x}\,.\,264.$$

$$\therefore\ x=62{\cdot}14\ldots\text{ feet.}$$

91. If v is the volume of the hydrometer at first, and v' after expansion, and if ρ and ρ' are the densities of the fluid,

$$\rho\,(v-\kappa x)=\rho'\,(v-\kappa x_1)=\rho'\,(v'-\kappa x_2),$$

$$\therefore\ v'-v=\kappa\,(x_2-x_1)\ \text{and}\ \left(\frac{\rho}{\rho'}-1\right)v=\kappa\,(x-x_1).$$

$$\therefore\ \frac{v'-v}{v}\ :\ \frac{1/\rho'-1/\rho}{1/\rho}\ ::\ x_2-x_1\ :\ x-x_1.$$

92. The resultant pressure of the liquid is in the vertical through the centre of the hemisphere, and, taking moments about the centre

$$\tfrac{3}{8}a \sin\theta \,.\, W = c\cos\theta \,.\, w.$$

93. Let U and V be the volumes of the hemisphere and the cylinder, W the weight of the float.

Then taking the hemisphere as the lowest portion,

$$W = (U+V)\,w = U\,.\,3w = \left(U + \frac{x}{h}V\right)2w.$$

$$\therefore\ x : h :: 1 : 4.$$

94. Let p be the pressure of the air forced in,

t the tension in the material of the tube,

$\frac{t}{p}$ is the radius of the portions which are not in contact with the sides of the triangle.

$\therefore\ \frac{2\pi t}{p}$ is their length.

If a be the length of the side of the triangle, the portion in contact with each side is $a - 2\sqrt{3}\,\frac{t}{p}$.

The original circumference of the tube was $\frac{\pi a}{\sqrt{3}}$,

$$\therefore\ 3a + t\left[\frac{2\pi}{p} - \frac{6\sqrt{3}}{p}\right] = \frac{\pi a}{\sqrt{3}}\left[1 + \frac{t}{\lambda}\right],$$

whence t is determined and thus the other quantities required.

95. Let l be the length of the faulty barometer,

γ the true reading when its reading is c.

A length $l - a$ of the tube is filled with air at a pressure $\alpha - a$.

A length $l - b$ of the tube is filled with air at a pressure $\beta - b$.

A length $l - c$ of the tube is filled with air at a pressure $\gamma - c$.

$$\therefore\ (l-a)(\alpha - a) = (l-b)(\beta - b) = (l-c)(\gamma - c),$$

$$l(\alpha - \beta - a + b) = a(\alpha - a) - b(\beta - b).$$

$$\therefore\ l - a = \frac{(a-b)(\beta - b)}{\alpha - a - (\beta - b)},$$

$$l - c = \frac{(a-c)(\alpha - a) - (b-c)(\beta - b)}{\alpha - a - (\beta - b)},$$

and
$$\gamma - c = \frac{(l-a)(\alpha - a)}{l - c} = \frac{(\alpha - a)(\beta - b)(a-b)}{(a-c)(\alpha - a) - (b-c)(\beta - b)}.$$

96. See Chap. VIII. Ex. 9.

97. Any such area as $A_n A_1 A_2 A_3$ must be a maximum consistently with keeping its sides constant.

$\therefore$ it must be inscribable in a circle, and hence the whole polygon is inscribable in a circle*.

If R be the radius of this circle,

$$R = \frac{c_1}{\sin \alpha_1} = \frac{c_2}{\sin \alpha_2} = \frac{c_3}{\sin \alpha_3} = \ldots\ldots$$

This problem however can be solved without assuming the property of the maximum area.

The fluid pressure being normal to the surfaces, and the same at all points of the same horizontal plane, the problem at once resolves itself into the equilibrium of a polygon of jointed rods, in one plane, the rods being acted upon, outwards, by normal forces at their middle points, proportional to their lengths.

Consider the equilibrium of four of the rods forming the polygon $A_1 A_2 A_3 A_4$. Taking any one rod $A_1 A_2$, the resultant of the stresses at A_1 and A_2 must bisect the rod at right angles.

These stresses are therefore equal, and consequently it follows that the stresses at all the joints are the same.

Let θ be the inclination to $A_2 A_1$ of the stress at A_2 or A_1, ϕ the inclination to $A_2 A_3$ of the stress at A_2 or A_3, and ψ the inclination to A_3 of the stress at A_3 or A_4.

Then, pr being the force on a rod of length r, and R the stress at each joint,

$$p \,.\, A_1 A_2 = 2R \sin \theta, \; p \,.\, A_2 A_3 = 2R \sin \phi, \; p \,.\, A_3 A_4 = 2R \sin \psi.$$

$$\therefore \frac{A_1 A_2}{\sin \theta} = \frac{A_2 A_3}{\sin \phi} = \frac{A_3 A_4}{\sin \psi}.$$

Let the straight line through A_2 at right angles to the direction of R intersect in O the straight line bisecting $A_1 A_2$ at right angles, and in O' the straight line bisecting $A_2 A_3$ at right angles.

Then $\qquad A_2 O = \frac{1}{2} A_1 A_2 \operatorname{cosec} \theta = \frac{1}{2} A_2 A_3 \operatorname{cosec} \phi = A_2 O'$,

and therefore O and O' are coincident, and O is the centre of the circle passing through $A_1 A_2 A_3$.

Moreover, OA_2 and OA_3 are perpendicular to the directions of R at A_2 and A_3, so that these directions are tangents to the circle.

Again it can be shewn, in exactly the same manner, that the straight line bisecting $A_3 A_4$ at right angles passes through the point O.

* A simple proof of this well-known theorem is given, with the aid of Trigonometry, and without infinitesimal changes, by Mr R. Chartres in Vol. LVIII., page 35, of "Solutions from the Educational Times."

Hence it follows that A_1, A_2, A_3, A_4 are concyclic, and therefore that all the angular points of the polygon are concyclic.

Finally, each of the expressions $c_1 \operatorname{cosec} a_1$, $c_2 \operatorname{cosec} a_2$, &c., represents the diameter of the circle.

98. Taking the width of the bridge as the unit of length, let $AB = 2a$.

Then if W is the moveable load, and if γ is the depression caused by placing W at G,

$$W = 2a\gamma w.$$

Now shift W from G to C and let θ be the angular tilt; then

$$W \,.\, CG = 2w \,.\, \tfrac{1}{2}a^2\theta \,.\, \tfrac{2}{3}a,$$

and if γ is the rise of A, $\gamma = AG \,.\, \theta$.

$\therefore$ $CG \,.\, AG$ is constant, equal to λ say.

Similarly $BG \,.\, GD$ and $PG \,.\, GQ$ are each equal to the same constant.

In the last case, if ϕ is the angular tilt,

$$\gamma = QG \,.\, \phi.$$

The depression of R due to load at P

$$= QR \,.\, \phi = \frac{QR}{QG} \cdot \gamma = \frac{\gamma}{\lambda} QR \,.\, PG = \frac{\gamma}{\lambda}(QG + GR)\, PG = \gamma + \frac{\gamma}{\lambda} GR \,.\, PG.$$

When load is placed at R, let Q' be the point which is unaltered in level, and let ψ be the angular tilt; then $\gamma = Q'G \,.\, \psi$, and $Q'G \,.\, RG = \lambda$.

$$\therefore \quad \text{depression of } P = PQ' \,.\, \psi = \frac{PQ'}{Q'G} \cdot \gamma = \frac{\gamma}{\lambda} PQ' \,.\, RG$$

$$= \gamma + \frac{\gamma}{\lambda} GR \,.\, PG.$$

99. Let S be the area of the shell, A that of the elliptic lamina.

Then $$S \sin a = A \sin \theta,$$

since the projection of each on a plane perpendicular to the axis of the cone must be the same.

If the rim be divided into small parts and the points of division joined to the vertex and the point where the axis cuts the base, the small triangles so formed are not only in the constant ratio of $\sin \theta$ to $\sin a$, but have their centres of gravity at the same depth. Hence the c.g. of the shell and lamina being at the same depth, the whole pressures vary as their areas, i.e. are as $\sin \theta$ to $\sin a$.

Let h be the length of the axis cut off by a plane through the point of the rim nearest the vertex perpendicular to the axis, $2a$ the major-axis of the lamina.

$$a \sin \theta - h \tan a$$

= distance of the c.g. of the shell or lamina from the axis.

$\therefore$ if W be the weight of shell and lamina, w that of the attached particle, the shell will float with axis vertical if

$$w \,.\, h \tan a = W \,.\, (a \sin \theta - h \tan a).$$

Now
$$\frac{2a}{h \sec a} = \frac{\sin 2a}{\sin(\theta - a)},$$

or
$$\frac{h}{a} = \sin\theta \cot a - \cos\theta.$$

$\therefore$ we require that

$$w(\sin\theta - \cos\theta \tan a) = W[\cos\theta \tan a],$$

or
$$\frac{w}{W} = \frac{\tan a}{\tan\theta - \tan a}.$$

If the liquid be sufficiently dense, it is clear that the rim may be entirely out of the liquid and in that case the c.g. of the displaced liquid is in the axis of the cone.

100. If $2a$ be the angle between the rods, w', w the intrinsic weights of the rods and water, θ the small angular displacement,

$$aw' = cw.$$

The length of one rod immersed increases to

$$c \cos a \sec(a + \theta) \text{ or } c(1 + \theta \tan a).$$

The moment about A of the liquid pressure upon it

$$= \tfrac{1}{2} wc^2 (1 + \theta \tan a)^2 \sin(a + \theta)$$
$$= \tfrac{1}{2} wc^2 \{\sin a + \theta(\sec a + \tan a \sin a)\}.$$

Putting $-\theta$ for θ and subtracting, the resultant moment of the liquid pressure on the two rods

$$= wc^2\theta (\sec a + \tan a \sin a).$$

The resultant moment of the weights of the two rods

$$= \tfrac{1}{2} w'a \,.\, a \sin(a + \theta) - \tfrac{1}{2} w'a \,.\, a \sin(a - \theta) = w'a^2 \cos a \,.\, \theta.$$

$\therefore$ the equilibrium is stable if

$$wc^2 (\sec a + \tan a \sin a) > w'a^2 \cos a,$$

or
$$c(\sec a + \tan a \sin a) > a \cos a,$$

or
$$c(3 - \cos 2a) > a(1 + \cos 2a).$$

CHAPTER XIII.

EXAMPLES.

1. (1) THE pressure on a square foot = the weight of 100 cubic feet of water

$$= 6250 \text{ lbs. weight} = 201{,}250 \text{ poundals.}$$

The pressure on a square centimetre = the weight of 3047·97 cubic centimetres of water

$$= 3047{\cdot}97 \text{ grammes weight} = 2{,}990{,}058{\cdot}57 \text{ dynes.}$$

(2) There is an addition to the pressure per square foot of the weight of 33 cubic feet of water

$$= 2062{\cdot}5 \text{ lbs. weight} = 66{,}412{\cdot}5 \text{ poundals.}$$

$\therefore$ Total pressure = 267,662·5 poundals.

On a square centimetre there is an additional pressure of 33 per cent. = 986,719·3281 dynes.

$\therefore$ Total pressure = 3,976,777·8981 dynes.

2. The pressure on the base = weight of half a cubic foot of olive oil + weight of half a cubic foot of alcohol

$$= \tfrac{1}{2}[{\cdot}9 + {\cdot}8] \times \text{weight of a cubic foot of water}$$

$$= {\cdot}85 \times 2012{\cdot}5 \text{ poundals} = 1710{\cdot}625 \text{ poundals.}$$

On the upper half of a side the pressure is equal to the weight of a column of alcohol 3 inches in depth and half a square foot in area, i.e.

$$= \tfrac{1}{8} \times {\cdot}8 \times 2012{\cdot}5 \text{ poundals} = 201{\cdot}25 \text{ poundals.}$$

On the lower half of a side the pressure is equal to the weight of a column of alcohol 6 inches high and of olive oil 3 inches high on a base of half a square foot

$$= [\tfrac{1}{4} \times {\cdot}8 + \tfrac{1}{8} \times {\cdot}9] \times 2012{\cdot}5 \text{ poundals}$$

$$= {\cdot}3125 \times 2012{\cdot}5 \text{ poundals} = 629{\cdot}00625 \text{ poundals.}$$

$\therefore$ Total pressure on a side = 830·25625 poundals.

3. $14\frac{1}{2}$ lbs. weight per square inch

$$=14\tfrac{1}{2}\div\cdot0022 \text{ grammes weight on } \left(\frac{1}{\cdot3937}\right)^2 \text{ sq. cm.}$$

$$=\frac{29\times(\cdot3937)^2\times981}{2\times\cdot0022} \text{ dynes per sq. cm.}$$

$$=1{,}002{,}178\cdot677456\dot{8}\dot{1} \text{ dynes per sq. cm.}$$

4. Resultant horizontal pressure on the curved surface = that on the plane surface = weight of a cylinder of the fluid of height 10 cm. and base 100π sq. cm.

$$=13\cdot568\times1000\pi\times981 \text{ dynes}$$

$$=41{,}815{,}349\cdot4528 \text{ dynes (taking } \pi=3\cdot1416).$$

Resultant vertical pressure

$$=\text{weight of } \frac{2}{3}\pi\times1000 \text{ cub. cm. of the liquid}$$

$$=13\cdot568\times\frac{2000\pi}{3}\times981 \text{ dynes}=27{,}877{,}232\cdot9685\dot{3} \text{ dynes.}$$

5. The new pressure $=1728\times15$ lbs. weight per sq. inch

$$=834{,}624 \text{ poundals per sq. inch.}$$

Now $\quad 1 \text{ cm.}=\cdot3937$ inch.

$\therefore\ 1$ sq. cm. $=\cdot155$ sq. inch (nearly).

$\therefore$ This pressure

$$=834{,}624\times13825\times\cdot155 \text{ dynes per sq. cm.}$$

$$=1{,}788{,}494{,}904 \text{ dynes per sq. cm.}$$

6. Pressure of atmosphere on a sq. inch

$$=\text{weight of 30 cub. inches of mercury}$$

$$=\frac{13\cdot568\times30\times2012\cdot5}{1728} \text{ poundals}=474\cdot0\dot{5} \text{ poundals.}$$

Height of barometer $=2\frac{1}{2}\times30\cdot4797$ cm.

$$=76\cdot2 \text{ cm. (q. p.).}$$

Pressure on a sq. cm. = weight of $76\cdot2$ cub. cm. of mercury

$$=76\cdot2\times13\cdot568\times981 \text{ dynes}=1{,}014{,}237\cdot85 \text{ dynes (q. p.).}$$

7. The internal pressure exceeds the external pressure by $6\cdot56$ lbs. weight per sq. inch

$$=211\cdot232 \text{ poundals per sq. inch.}$$

$\therefore$ Tension per linear inch of the curved surface of the cylinder

$$=211\cdot232r \text{ poundals,}$$

r being the radius of the cylinder.

8. The atmospheric pressure is one megadyne per sq. cm. (vide Errata in Treatise).

The excesses of internal pressure over that of the atmosphere are:—

$$\text{(i)}\quad \frac{2\times 80}{2}=80 \text{ dynes per sq. cm.}$$

$$\text{(ii)}\quad \frac{2\times 30}{2\cdot 5}=24 \text{ dynes per sq. cm.}$$

These are in the ratio 10 : 3.

The masses are proportional to the products of the pressure and volume, and are therefore in the ratio

$$1{,}000{,}080\times 8 : 1{,}000{,}024\times 15\cdot 625$$
$$=1{,}600{,}128 : 3{,}125{,}075=1 : 2 \text{ nearly.}$$

9. When floating, a length $\sigma l/\rho$ feet is immersed. When totally immersed, the force required to hold it down exceeds its weight by the weight of a volume

$$\frac{\rho-\sigma}{\rho} l\kappa \text{ of water,}$$

i.e. by $$(\rho-\sigma)\,g\kappa l \text{ poundals.}$$

The average force exerted in depressing it is one-half of this amount.

$$\therefore \text{ Work done}=\frac{1}{2}(\rho-\sigma)\,g\kappa l\times\frac{\rho-\sigma}{\rho}l$$
$$=\frac{1}{2}g\kappa l^2\frac{(\rho-\sigma)^2}{\rho} \text{ foot-poundals.}$$

10. In this case, let x be the distance through which the block must be depressed, in order to be wholly immersed.

Then $\frac{\rho-\sigma}{\rho}l-x$ is the rise of the water-surface.

$$\therefore\quad \kappa x=(\kappa'-\kappa)\left\{\frac{\rho-\sigma}{\rho}l-x\right\},$$

or $$x=\left(1-\frac{\kappa}{\kappa'}\right)\frac{\rho-\sigma}{\rho}l.$$

$$\therefore \text{ Work done}=\frac{1}{2}g\kappa l^2\left(1-\frac{\kappa}{\kappa'}\right)\frac{(\rho-\sigma)^2}{\rho} \text{ foot-poundals,}$$

the average force exerted being the same as in (9).

11. Force required to raise the block so long as it is totally immersed $=g(\sigma-\rho)\,\kappa l$ poundals.

$\therefore$ Work done in lifting the block till its upper surface is in the surface of the water

$$=g(\sigma-\rho)\,\kappa l\,(h-l) \text{ foot-poundals.}$$

In raising it out of the water, the work done is

$$\tfrac{1}{2}\{g\sigma kl+g(\sigma-\rho)\,kl\}\,l \text{ foot-poundals.}$$

$$\therefore \text{ Total work done}=g(\sigma-\rho)\,klh+\tfrac{1}{2}g\rho kl^2 \text{ foot-poundals.}$$

CHAPTER XIV.

EXAMPLES.

1. WE have (see Art. 200)

$$p=\rho\left[\tfrac{1}{2}\omega^2 QN^2 - g\,.\,ON\right];$$

$\therefore$ when ON remains constant, the difference of pressure varies as the difference of the values of QN^2.

2. Let l be the latus rectum of the vessel,

$$\frac{2g}{\omega^2} \text{ is that of the liquid surface.}$$

The volume of a paraboloid being one-half that of the cylinder on the same base and of the same height, the surface of the fluid must bisect the axis of the vessel.

$\therefore$ r being the radius of the rim,

$$\frac{r^2}{l}=2\,.\,\frac{r^2}{2g/\omega^2};$$

$$\therefore\ \omega^2=g/l.$$

3. Let ω be the angular velocity of rotation,

h the height, r the radius of the cylinder,

x the depth immersed,

σ, ρ the densities of solid and fluid.

$$\sigma h=\rho\left[x-\tfrac{1}{2}\frac{\omega^2 r^2}{2g}\right];$$

$$\therefore\ x=\frac{\sigma}{\rho}h+\frac{\omega^2 r^2}{4g}.$$

4. The common surface must be a surface of equal pressure in both liquids, which is possible, since the latera recta of the surfaces of equilibrium are independent of the densities of the fluids.

The surface is $\therefore$ a paraboloid of revolution.

5. If the paraboloidal surface touches the surface of the cone at the rim,

$$h^2 \tan^2 a = \frac{2g}{\omega^2} \cdot \frac{h}{2} \text{ and } \therefore \omega^2 = \frac{g}{h} \cot^2 a.$$

If $\omega < \cot a \sqrt{\frac{g}{h}}$, the depth of the vertex of the paraboloid below the rim of the cone

$$= r^2 \div \frac{2g}{\omega^2} = \frac{\omega^2 r^2}{2g}, \text{ where } r = h \tan a,$$

and the volume which runs over $= \frac{1}{2}\pi r^2 \cdot \frac{\omega^2 r^2}{2g}$.

If $\omega^2 = \frac{g}{h} \cot^2 a$, this is $\frac{1}{4}\pi h^3 \tan^2 a$.

If $\omega^2 > \frac{g}{h} \cot^2 a$, the liquid left in the cone will touch its surface in a circle of radius $2x \tan a$, such that

$$(2x \tan a)^2 = \frac{2g}{\omega^2} x.$$

Hence $x = g \cot^2 a / 2\omega^2$, and the volume of liquid left in the cone

$$= \tfrac{1}{3}\pi x^2 \tan^2 a \cdot 2x - \tfrac{1}{2}\pi x^3 \tan^2 a$$
$$= \frac{\pi}{48} \frac{g^3 \cot^4 a}{\omega^6}.$$

6. If x is the part of the axis not immersed, and a the radius of the bowl, the quantity which runs over is $\frac{1}{2}\pi a^2 x$, where $x = \frac{a^2\omega^2}{2g}$, and is therefore $\frac{\pi\omega^2 a^4}{4g}$.

7. Let P, Q be the free surfaces of the liquid in the tube, ANM the axis of rotation, C the centre of the ellipse. Then PQ passes through C, the tube being half full.

Draw PN, QM, CR, perpendicular to the axis of rotation

$$PN^2 = \frac{2g}{\omega^2} \cdot AN, \qquad QM^2 = \frac{2g}{\omega^2} \cdot AM;$$

$$\therefore \tan\theta = \frac{QM - PN}{AM - AN} = \frac{g}{\omega^2 \cdot CR} = \frac{g}{\omega^2 p}.$$

8. If we imagine the cylinder extended and the free surface completed to the curved surface of the cylinder, we see that the pressure on the upper end is equal to the weight of a volume

$$\tfrac{1}{2}\pi a^2 \cdot \frac{\omega^2 a^2}{2g} \text{ of liquid,}$$

i.e. it is $$g\rho\pi a^2 \cdot \frac{a^2\omega^2}{4g} \text{ poundals.}$$

That on the lower end $=g\rho\pi a^2\left[\frac{a^2\omega^2}{4g}+h\right]$.

The whole pressure on the curved surface

$$=\tfrac{1}{2}g\rho\left[h+\frac{\omega^2a^2}{2g}\right].2\pi a\left(h+\frac{\omega^2a^2}{2g}\right)-\tfrac{1}{2}g\rho.\frac{\omega^2a^2}{2g}.2\pi a.\frac{\omega^2a^2}{2g}$$

$$=g\rho\pi ah\left(h+\frac{\omega^2a^2}{g}\right).$$

9. If W be the weight of the bowl, a its radius,

W together with the weight of fluid in the bowl must be equal to the weight of fluid in a cylinder on the same base as the hemisphere and having a length a of its axis immersed in rotating fluid.

$$\therefore\ W+\tfrac{2}{3}g\rho\pi a^3=g\rho\pi a^2\left(a+\frac{a^2\omega^2}{4g}\right);$$

$$\therefore\ W=g\rho\pi a^3\left(\frac{a}{3}+\frac{a^2\omega^2}{4g}\right).$$

10. Let r be the distance of the cork from the axis, w, W the weights of the cork and of the water it displaces. The fluid pressure is equivalent to a force vertically upwards, equal to W, and a force to the axis equal to $\frac{W}{g}\omega^2r$.

Let T be tension of string, and θ its inclination to the vertical.

Then $\qquad T\cos\theta=W-w,\quad$ and $\frac{W}{g}\omega^2r-T\sin\theta=\frac{w}{g}\omega^2r,$

so that $\qquad \frac{W}{g}\omega^2r\cos\theta=\frac{w}{g}\omega^2r\cos\theta+(W-w)\sin\theta.$

If a be the radius of the cylinder, and l the length of the string,

$$a=r+l\sin\theta.$$

These equations determine r and θ.

11. ω being the required velocity,

$$\tfrac{1}{2}h.\pi a^2=\tfrac{1}{2}\frac{\omega^2a^2}{2g}.\pi a^2;$$

$$\therefore\ \omega=\sqrt{2gh}/a.$$

12. In this case we have

$$\tfrac{1}{2}\frac{\omega^2h^2}{6g}.\pi.\frac{h^2}{3}=\tfrac{1}{2}.\frac{h}{3}.\pi.\frac{h^2}{3};$$

$$\therefore\ \omega=\sqrt{\frac{2g}{h}}.$$

13. If $\frac{a^2}{4} \not> \frac{2g}{\omega^2} a$, i.e. $\omega \not> 2\sqrt{2g/a}$, the free surface does not intersect the middle tube and $\therefore$ no liquid flows out.

When $\omega = 2\sqrt{2g/a}$ the vertex of the paraboloidal free surface is at the point of intersection of the axis of rotation and the middle tube.

The whole pressure on tube at rest

$$= \tfrac{2}{3} g\rho a \times \text{surface of tube}.$$

The pressure on the middle tube when rotating is $\frac{1}{3}$ of what it was before, since the area of the segment of the parabola $= \frac{2}{3}$ area of circumscribing square.

$$\therefore \text{ whole pressure on tube} = \tfrac{4}{3} g\rho a \times \text{surface of tube}.$$

14. Let AB be a section of the surface of equal pressure which cuts the sphere (centre O) at right angles at P. OAN being the axis of rotation

$$PN^2 = \frac{2g}{\omega^2} AN = 3c \,.\, AN;$$

$$\therefore \; c^2 \sin^2\theta = 3c \,.\, \frac{c\cos\theta}{2}.$$

The real root of which is given by $\cos\theta = \frac{1}{2}$.

Pressure at $P =$ pressure at $A = g\rho \,.\, AE$

$$= g\rho\left(c - \frac{c\cos\theta}{2}\right) = \tfrac{3}{4} g\rho c.$$

15. Suppose the free surface continued above the lid; volume of fluid above lid

$$= \tfrac{1}{2}\pi a^2 \,.\, \frac{\omega^2 a^2}{2g} - \tfrac{1}{2}\pi b^2 \,.\, \frac{\omega^2 b^2}{2g} - \pi(a^2 - b^2)\frac{\omega^2 b^2}{2g} = \frac{\pi\omega^2}{4g}(a^2 - b^2)^2.$$

The weight of this is the upward pressure on the lid, and the centre of pressure is the centre of the lid, i.e. its c.g. The whole volume of fluid in the cylinder

$$= \pi a^2 h - \tfrac{1}{2}\pi b^2 \frac{\omega^2 b^2}{2g} = \frac{\pi\omega^2}{4g}(a^4 - b^4);$$

$$\therefore \text{ weight of lid : weight of fluid} = a^2 - b^2 : a^2 + b^2.$$

16. Replace the spheres by vertical columns of liquid of length b.

The free paraboloidal surface will pass through the tops of these columns, and intersect the axis at the depth x below the centre, such that $a^2\omega^2 = 2g(b + x)$.

The pressure in the tube at the depth y below the centre

$$=\rho\{\tfrac{1}{2}\omega^2(a^2-y^2)+g(y-x)\},$$

$$=\frac{\rho}{2}\left\{2b-\left(\omega y-\frac{g}{\omega}\right)^2\right\},$$

which is greatest when $y=g\omega^{-2}$.

17. If T is the centre of the face, we have to find the point of contact with the face of a paraboloid of latus rectum $2g/\omega^2$, the axis of which is in the vertical through T. If P is the point of contact and PN its ordinate, and A the vertex of the paraboloid,

$$PN=2AN\cot a,\quad PN^2=\frac{2g}{\omega^2}AN;$$

$$\therefore\ PN=\frac{g}{\omega^2}\tan a,\text{ and } PT=\frac{g}{\omega^2}\frac{\sin a}{\cos^2 a}.$$

18. Since the volume of the paraboloid is one-half that of a cylinder of the same base and height, the depth of the vertex of the paraboloid when the water reaches the rim is $\frac{2h}{n}$,

$$\therefore\ a^2=\frac{2g}{\omega^2}\cdot\frac{2h}{n}.$$

19. The depth of the vertex of the paraboloidal surface when the water has ceased to flow over is $\frac{r^2\omega^2}{2g}$.

The volume of water which has run over is $\therefore$

$$\tfrac{1}{2}\pi r^2\cdot\frac{r^2\omega^2}{2g}=\tfrac{1}{4}\pi r^4\omega^2/g$$

$$=\tfrac{3}{8}\frac{r\omega^2}{g}\times\text{volume of hemisphere.}$$

$\therefore$ the pressure on the table : original weight of liquid

$$=1-\tfrac{3}{8}\frac{r\omega^2}{g}:1=8g-3\omega^2r:8g.$$

20. The free surface would cut the cylinder at a distance $\frac{r^2\omega^2}{2g}$ above the top;

$\therefore$ the pressure on the top is

$$\frac{a}{\pi}\cdot\pi r^2\cdot\frac{r^2\omega^2}{4g}=\frac{a}{4}\frac{r^4\omega^2}{g},$$

$2a$ being the angle of the wedge.

21. We have to find the weight of the liquid which would fill the space between the cone, vertical lines through all points of its base, and a paraboloidal surface of latus rectum $2g/\omega^2$ and vertex at the vertex of the cone.

If k be the height of such a surface above the vertex, $h^2 \tan^2 a = \frac{2g}{\omega^2} k$.

The required weight : that of the fluid in the cone as

$$\frac{2h}{3} + \frac{k}{2} : \frac{h}{3},$$

$$\therefore k = \tfrac{2}{3} h, \text{ and } 3h\omega^2 = 4g \cot^2 a.$$

22. If P is a point in the free surface we find, by considering the motion of an element as in Art. 199, that

$$\omega^2 . NG \text{ is constant.}$$

If $$\omega . PN \propto \frac{1}{\omega}, \omega^2 PN \text{ is constant,}$$

$$\therefore PN \propto NG,$$

$\therefore$ the surface is that of a right circular cone.

CHAPTER XV.

EXAMPLES.

1. IF the required point divide the side of the cone in the ratio $x : 1-x$, h being the height, $2a$ the angle of the cone,

the water issues with a vertical velocity upwards $\sqrt{2ghx}\sin a$ and a horizontal velocity $\sqrt{2ghx}\cos a$.

If it is to fall just outside the base, the time of flight must be

$$\frac{(1-x)\,h\tan a}{\sqrt{2ghx}\cos a},$$

$\therefore$ $(1-x)\,h =$ the vertical space described in this time,

$$=\tfrac{1}{2}g\,\frac{(1-x)^2h^2\tan^2 a}{2ghx\cos^2 a}-\sqrt{2ghx}\sin a\,.\,\frac{(1-x)\,h\tan a}{\sqrt{2ghx}\cos a};$$

$$\therefore\ 1=\frac{(1-x)\tan^2 a}{4x\cos^2 a}-\tan^2 a,$$

or

$$(1-x)\tan^2 a=4x.$$

$$\therefore\ x=\frac{\tan^2 a}{4+\tan^2 a}.$$

2. Taking the case of a hole at an angular distance θ from the highest point of the circle.

Its depth is $a(1-\cos\theta)$.

$\therefore$ the time of falling to the level of the lowest point is

$$\sqrt{\frac{2a(1+\cos\theta)}{g}}.$$

$\therefore$ the distance from the plane face of the point where the fluid from this hole meets the horizontal plane through the lowest point is

$$\sqrt{2ga(1-\cos\theta)}\,.\,\sqrt{\frac{2a}{g}(1+\cos\theta)}=2a\sin\theta.$$

The horizontal distance from the centre is $a\sin\theta$.

$\therefore$ the point lies on a line through the lowest point inclined to the vertical face at an angle $\tan^{-1} 2$, i.e. the trace on the plane is two straight lines.

3. If h be the height of a cylinder, a its radius, ρ the density of the water in it, w the weight of that water,

$$w = \rho g \,.\, \pi a^2 \,.\, h.$$

The pressure on the curved surface

$$= 2\pi a \,.\, h \,.\, \rho g \,.\, \frac{h}{2} = \rho g \pi a h^2 = \frac{h}{a} w.$$

If the downward acceleration be f, the pressures are all diminished in the ratio $g : g - f$.

If the acceleration f be upwards they are increased in the ratio

$$g : g + f.$$

$\therefore$ the pressures are $\dfrac{g+f}{g} w \,.\, \dfrac{h}{a}$ and $\dfrac{g-f}{g} w' \dfrac{h'}{a'}$.

And $$f = \frac{w' - w}{w' + w} g.$$

$$\therefore \frac{g-f}{g} = \frac{2w}{w' + w} \text{ and } \frac{g+f}{g} = \frac{2w'}{w' + w};$$

$\therefore$ the pressures are as $\dfrac{h}{a} : \dfrac{h'}{a'}$.

If the cylinders be similar the pressures are equal, if they be of equal height, the pressures are in the inverse ratio of the radii.

4. If W is given weight, W' weight of cone, and w of water,

$$\text{acceleration} = f = \frac{W - W' - w}{W + W' + w} g.$$

Hence $\dfrac{w}{g} \,.\, f =$ upward force on the water,

$=$ resultant vertical pressure $- w$,

and whole pressure $= \operatorname{cosec} \alpha$ (resultant pressure).

5. If h, h' be the heights of the cones,

α, β the areas of their bases,

the weights of the contained water are as $\alpha h : \beta h'$.

$$\therefore \text{ the acceleration is } \frac{\alpha h - \beta h'}{\alpha h + \beta h'} g = f.$$

The pressure on the bases are as

$$\frac{g-f}{g} \alpha h : \frac{g+f}{g} \beta h',$$

which are always equal.

The weights in the second case are as

$$\alpha h(1-m^3) : \beta h'(1-n^3).$$

The pressures on the bases are as

$$(g-f)\,\alpha h(1-m) : (g+f)\,\beta h'(1-n),$$

where
$$\frac{f}{g}=\frac{\alpha h(1-m^3)-\beta h'(1-n^3)}{\alpha h(1-m^3)+\beta h'(1-n^3)};$$

∴ the pressures are as

$$(1-n^3)(1-m) : (1-m^3)(1-n),$$

i.e. as
$$1+n+n^2 : 1+m+m^2.$$

6. Let w, w' be the intrinsic weights of the water and the material of the shells, $a\,.\,b$ their radii, λa, λb their thicknesses.

The weights of the shells are

$$w'\,.\,4\pi a^2\,.\,\lambda a\,.=4\pi\lambda w'a^3$$

and
$$4\pi\lambda w'b^3.$$

The contained fluids weigh $\frac{4}{3}\pi wa^3$, $\frac{4}{3}\pi wb^3$ respectively

$$\therefore \text{ the acceleration}=\frac{a^3-b^3}{a^3+b^3}\cdot g.$$

The resultant pressure on the larger shell

$$=\tfrac{4}{3}\pi wa^3\left(1-\frac{a^3-b^3}{a^3+b^3}\right)=\tfrac{8}{3}\pi w\frac{a^3b^3}{a^3+b^3},$$

and that on the smaller

$$=\tfrac{4}{3}\pi wb^3\left(1+\frac{a^3-b^3}{a^3+b^3}\right)=\tfrac{8}{3}\pi w\frac{a^3b^3}{a^3+b^3}.$$

7. If ρ, σ are the densities of lead and water, the acceleration in the water is $\frac{\rho-\sigma}{\rho}g$;

∴ if V is the velocity on reaching the water, v the velocity at the depth x,

$$v^2=V^2+2\frac{\rho-\sigma}{\rho}gx.$$

8. If nk, k, be the areas of the pipes, the energy expended per unit of time is

$$\tfrac{1}{2}\rho\,.\,nkV\,.\,V^2=\tfrac{1}{2}\rho knV^3,$$

ρ being the mass of unit volume of water.

The work done is to raise a column of height h with velocity v, and is therefore $\rho gkhv$ per unit of time.

Equating these we obtain

$$h=nV^3/2gv.$$

9. If F be the friction when v is the velocity and a the area of wetted surface,

$$F = k \cdot av^2.$$

When $v = 12$ and $a = 1$, $F = 1$, $\therefore\ k = \frac{1}{144}$.

Thus when $v = 3$, $a = \pi \times \frac{7}{12} \times 5280$,

$$F = \frac{1}{144} \times 7 \times 440\pi \times 9 = \frac{385\pi}{2} \text{ lbs. wt.}$$

The work done against friction is $\frac{385\pi}{2} \times 3$ ft. lbs. per sec.

The loss of H. P. is $\frac{385\pi}{2} \cdot \frac{3}{550} = \frac{21}{20}\pi = 3{\cdot}3$ H. P., q. p.

10. A volume $2nAl$ is pumped out per minute.

$\therefore\ \frac{2nAl}{B}$ is the velocity per minute with which it issues.

11. The accelerations down the plane in the two cases are

$$g(\sin a + \mu \cos a), \quad g(\sin a - \mu \cos a).$$

In the first case let the normal to the free surface be inclined to the plane at the angle θ.

Then since the resultant fluid pressure R on an element m in the surface and the weight mg have for their resultant the force

$$mg(\sin a + \mu \cos a)$$

parallel to the plane, it follows that

$$R \cos \theta + mg \sin a = mg(\sin a + \mu \cos a),$$

and

$$R \sin \theta - mg \cos a = 0,$$

$\therefore\ \tan \theta = \mu^{-1} = \cot \lambda$, if λ is the angle of friction.

Similarly, in the second case, $\tan \phi = -\cot \lambda$;

$\therefore$ the angle between the two directions is 2λ.

12. Let f_1 be the acceleration of the train when ascending and f_2 when descending.

Then if Mf is the resultant of the pull of the engine and the resistance,

$$Mf_1 = Mf - Mg \sin a, \quad Mf_2 = Mf + Mg \sin a.$$

Then, if θ is the inclination to the vertical of the normal to the free surface when the train is ascending,

$$\tan \theta (g + f_1 \sin a) = f_1 \cos a \text{ (Art. 212).}$$

If ϕ is the inclination when the train is descending it follows, by the same process of reasoning as that of Art. 212, that

$$\tan \phi (g - f_2 \sin a) = f_2 \cos a.$$

We hence obtain

$$\tan\theta=\frac{\cos\alpha\,(f-g\sin\alpha)}{g\cos^2\alpha+f\sin\alpha},\quad \tan\phi=\frac{\cos\alpha\,(f+g\sin\alpha)}{g\cos^2\alpha-f\sin\alpha},$$

and it follows that

$$\tan(\phi-\theta)=\tan 2\alpha.$$

13. If f is the acceleration we have as in Ex. 12,

$$\tan\theta\,(g+f\sin\alpha)=f\cos\alpha,\quad \tan\phi\,(g-f\sin\beta)=f\cos\beta.$$

Now
$$\frac{f}{g}=\frac{M'\sin\beta-M\sin\alpha}{M'+M},$$

$$\therefore\ \text{if } M' : M=\sec\beta : \sec\alpha,$$

$$\frac{f}{g}=\frac{\sin\frac{1}{2}(\beta-\alpha)}{\cos\frac{1}{2}(\beta+\alpha)},$$

and we then find that

$$\tan\theta=\tan\phi=\tan\frac{\beta-\alpha}{2}.$$

14. The velocity of the particle on reaching the fluid is $\sqrt{2ga\cos\alpha}$, a being the radius of the axis of the tube.

Let m, M be the masses of the particle and fluid, which we shall suppose inelastic, u the velocity acquired by the fluid.

Then
$$(m+M)\,u=m\sqrt{2ga\cos\alpha},$$

$$\therefore\ u=\frac{m}{m+M}\sqrt{2ga\cos\alpha}.$$

At a distance θ from the particle, the impulsive pressure is such as to produce the velocity u in the portion beyond that point;

$$\therefore\ \text{the impulse}\quad=\frac{2a-\theta}{2a}Mu=\left(1-\frac{\theta}{2a}\right)\frac{Mm}{M+m}\sqrt{2ga\cos\alpha}.$$

15. Let h be the height, r the radius of the cylinder.

The impulsive pressure at a depth x is such as to destroy a velocity v in a column of fluid of height x and $\therefore\ =\rho x\,.\,v$.

The whole impulse on the curved surface

$$=v\rho\,.\,2\pi rh\,.\,\frac{h}{2}=v\rho\pi rh^2.$$

16. The impulse at the depth x below the vertex is ρvx.

If h is the height of the cone, the resultant impulse on the base $=\rho v\pi h^3\tan^2\alpha$, and the whole impulse on the curved surface

$$=\rho v\,.\,\pi h^2\tan^2\alpha\operatorname{cosec}\alpha\,.\,\tfrac{2}{3}h.\quad\text{(Art. 215.)}$$

If t is the impulsive tension at the depth x in direction of the generating lines, $2\pi x\tan\alpha\,.\,t\cos\alpha=$ resultant vertical impulse on curved surface $=\frac{2}{3}\rho v\pi x^3\tan^2\alpha$.

17. The impulsive pressure on the cork is sufficient to cause an equal volume of water to move with the velocity imparted to the vessel. The cork being lighter than water, the velocity it acquires is greater than that of the vessel and the contained water.

18. If ψ is the deviation from the central radius of the normal to the surface of the water, which is flowing without any acceleration in the direction of its motion, then, as in Art. 205,

$$\sin\psi : \sin\frac{\pi}{4} :: w^2\frac{r}{\sqrt{2}} : g,$$

$$\therefore\ \psi = \frac{1}{2}\frac{\omega^2 r}{g} = \frac{2000\times 5280}{g}\left(\frac{2\pi}{24\times 60\times 60}\right)^2 = \tfrac{1}{570} \text{ nearly.}$$

19. The pressure of the pipe on the water is $\rho v^2/r$ per unit length, r being the radius of the circle.

$$\therefore \text{ the tension} = \text{pressure} \times \text{radius} = \rho v^2.$$

Since this is independent of the curvature, the second part follows immediately.

20. The momentum destroyed is that of the fluid as it enters the bucket diminished by its vertical momentum as it issues from the aperture in the base of the bucket.

The mass which falls into the bucket in one second is $\frac{1}{15}$ lb., and its velocity in feet per second

$$= \{900 + 16g\}^{\frac{1}{2}} = 37{\cdot}619,$$

the vertical velocity of the issuing fluid

$$= \sqrt{2g}\,.\,\frac{1}{\sqrt{2}} = 5{\cdot}674,$$

$\therefore$ the extra downward pressure, which is the rate of destruction of momentum,

$$= \tfrac{1}{15}(37{\cdot}619 - 5{\cdot}674) \text{ poundals} = 2{\cdot}129 \text{ poundals} = {\cdot}066 \text{ lbs. weight.}$$

MISCELLANEOUS PROBLEMS. II.

1. LET 2θ be the angle made by the rth bounding radius with the surface.

The area of the first r sectors is $a^2\theta$, and the depth of the centroid of the area is

$$a \sin^2 \theta/\theta.$$

$$\therefore\ a^3 \sin^2 \theta = \frac{r}{n} \cdot \frac{\pi a^2}{2} \cdot \frac{2a}{\pi}, \text{ or } \sin^2 \theta = \frac{r}{n}.$$

2. If h is the depth of the centre of the sphere below the surface, we have, from Art. 57,

$$X = w\pi a^2 h \cos \theta, \qquad Y - \tfrac{4}{3} w\pi a^3 = w\pi a^2 h \sin \theta,$$

and, measuring off from a fixed point two sides of a rectangle proportional to $\cos\theta$ and $\sin\theta$, the locus of the end of the diagonal is a circle.

3. Let the weight of the rod be W, that of the heavy particle mW, $2a$ the length of the rod.

The distance of the c. g. of the combination from the lower end of the rod is $a/(m+1)$.

The length immersed is $2 \cdot \dfrac{m+1}{n} a$.

$\therefore$ The c. g. of the displaced water is $\dfrac{m+1}{n} a$ from the lower end.

$\therefore$ The equilibrium is stable if

$$\frac{1}{m+1} < \frac{m+1}{n} \text{ or } m > \sqrt{n} - 1.$$

4. Let h be the height of each triangular face.

The pressure on each face is $2W \operatorname{cosec} \beta/n$.

The centre of pressure is distant $\frac{1}{4}h$ from the hinge.

The centre of gravity is distant $\frac{1}{3}h$ from the hinge.

$$\therefore\ \tfrac{1}{3} h \sin\beta \,.\, w \nless \frac{2W \operatorname{cosec}\beta}{n} \,.\, \tfrac{1}{4} h \text{ or } nw \nless \tfrac{3}{2} W \operatorname{cosec}^2 \beta.$$

The pressure on a face is obtained from the consideration of the fact that the vertical component of the n pressures on the faces is equal to the weight of water which would be contained between the pyramid and a cylinder on the same base and of the same height, i.e. to $2W$.

5. If v be the velocity of descent, d the depth of the centre of gravity, $d+x$ that of the centre of pressure when the area is just immersed. After a time t the c. g. is at a depth $d+vt$, the pressure is $wA(d+vt)$, A being the area.

If $d+x-z$ be now the depth of the centre of pressure,

$$wA(d+vt)(d+x-z)=wAd.x.$$

$$\therefore\ z=d+x-\frac{dx}{d+vt}=d+\frac{xvt}{d+vt}.$$

If u be the rate of change of z,

$$u\tau=\frac{xv(t+\tau)}{d+v(t+\tau)}-\frac{xvt}{d+vt}=\frac{xvd\tau}{(d+vt)(d+vt+v\tau)},$$

$$u=\frac{xvd}{(d+vt)^2}\text{ when }\tau\text{ is indefinitely diminished.}$$

6. Let $2x$ be the height of the bell.

Then $d+x$ is the depth of the surface of the water in the bell, which must be equal to h since the pressure within the bell is double the atmospheric pressure.

$$\therefore\ x=h-d.$$

7. Let P, Q be the areas of the triangles formed by joining the points whose depths are α, β, γ and δ, β, γ respectively.

Then $\qquad P.\frac{1}{3}(\alpha+\beta+\gamma)+Q.\frac{1}{3}(\delta+\beta+\gamma)=(P+Q)h.$

If x be the depth of the centre of pressure of the quadrilateral, then, from Art. 183,

$$\{P(\alpha+\beta+\gamma)+Q(\beta+\gamma+\delta)\}x$$
$$=P.\frac{\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta}{2}+Q.\frac{\beta^2+\gamma^2+\delta^2+\gamma\delta+\delta\beta+\beta\gamma}{2}.$$

Substituting for the ratio $P:Q$,

$$\begin{aligned}6hx(\alpha-\delta)&=(3h-\beta-\gamma-\delta)(\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta)\\&\quad+(\alpha+\beta+\gamma-3h)(\beta^2+\gamma^2+\delta^2+\gamma\delta+\delta\beta+\beta\gamma)\\&=(3h-\beta-\gamma)(\alpha-\delta)(\alpha+\beta+\gamma+\delta)+(\alpha-\delta)(\beta^2+\gamma^2+\beta\gamma-\alpha\delta).\end{aligned}$$

$$\therefore\ x=\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)-\frac{1}{6h}(\beta\gamma+\gamma\alpha+\alpha\beta+\alpha\delta+\beta\delta+\gamma\delta).$$

8. The pressure of the water on the face AC acts perpendicular to AC at a point which divides AC in the ratio 2 : 1.

$\therefore$ unless $\tan C < 2\tan A$, its direction cuts the base beyond B, and the maintenance of equilibrium depends on the weight of the prisms.

Equilibrium will be maintained, if this condition hold, whatever may be the density of the prisms, provided they do not slip.

The sectional area of the canal is $b^2 \sin C \cos C$.

The pressure on the base of each prism

$= w \,.\, \frac{1}{2} b \sin C\,[b\cos C + \rho a] = w \,.\, \frac{1}{2} b \sin C\,[(1+\rho)\, b \cos C + \rho c \cos B]$.

The horizontal pressure on each prism is

$$w \,.\, \tfrac{1}{2} b^2 \sin^2 C.$$

$\therefore$ the angle of friction must be greater than

$$\cot^{-1}\{(1+\rho)\cot C + \rho \cot B\}$$

(remembering that $b \sin C = c \sin B$).

9. The weight of the first cone is $w\pi a^2 \operatorname{cosec} a$, and the volume it encloses is $\frac{1}{3}\pi a^3 \cot a$.

Commencing with the smallest cone, let $\rho_1, \rho_2, \ldots \rho_n$ be the densities of the gases.

Then
$$\begin{aligned} w\pi a^2 \operatorname{cosec} a + 2\pi a\kappa &= k(\rho_1 - \rho_2)\, \pi a^2, \\ 4w\pi a^2 \operatorname{cosec} a + 4\pi a\kappa &= k(\rho_2 - \rho_3)\, 4\pi a^2, \\ \ldots \quad \ldots \quad \ldots & \quad \ldots \quad \ldots \\ n^2 w\pi a^2 \operatorname{cosec} a + 2n\pi a\kappa &= k\rho_n \,.\, n^2 \pi a^2. \end{aligned}$$

Multiplying these equations by 1, 2, 3,..., n and adding them together we obtain

$$w\pi a^2 \operatorname{cosec} a \left\{\frac{n(n+1)}{2}\right\}^2 + 2\pi a\kappa\, \frac{n(n+1)(2n+1)}{6}$$
$$= k\pi a^2 [\rho_1 + \rho_2 (2^3 - 1^3) + \ldots + \rho_n \{n^3 - (n-1)^3\}].$$

But, if $Vn^3\sigma$ is the total mass of gas,

$$Vn^3\sigma = V\rho_1 + V\rho_2(2^3 - 1^3) + \ldots + V\rho_n\{n^3 - (n-1)^3\},$$

$$\therefore\ wa \operatorname{cosec} a \left\{\frac{n(n+1)}{2}\right\}^2 + \tfrac{1}{3}\kappa n(n+1)(2n+1) = kan^3\sigma.$$

The upward force on the cone

$$= k\sigma\pi n^2 a^2 - w\pi n^2 a^2 \operatorname{cosec} a - 2\pi n a\kappa$$
$$= \tfrac{1}{12}\pi a(n-1)\{3wan(n-1)\operatorname{cosec} a + 4\kappa(2n-1)\}.$$

10. The area of the cross-section is $\left(\frac{\pi}{\sqrt{3}}+\frac{3}{8}\right)a^2$, and that of the immersed portion is $\frac{\sqrt{3}}{4}\pi a^2$,

$$\therefore\ \rho\,.\,\frac{\sqrt{3}}{4}\pi a^2=\sigma\,.\left(\frac{\pi}{\sqrt{3}}+\frac{3}{8}\right)a^2.$$

It will be found that the distance CG of the centroid of the cross-section from the centre is

$$3\sqrt{3}a/(8\pi+3\sqrt{3}).$$

If H is the centroid of the immersed portion,

$$CH=\frac{4a}{3\pi},$$

$$\therefore\ HG=\frac{4a}{3\pi}-\frac{3\sqrt{3}a}{8\pi+3\sqrt{3}}.$$

The radius of curvature r at H of the curve of buoyancy, which is a similar and similarly situated concentric ellipse

$$=\left(\frac{4b}{3\pi}\right)^2\Big/\frac{4a}{3\pi}=\frac{a}{\pi}.$$

Now $$HG<r,\ \text{if}\ \frac{a}{3\pi}<\frac{3\sqrt{3}a}{8\pi+3\sqrt{3}},$$
which is the case.

The equilibrium in the symmetrical position is therefore stable.

11. Let x be the length of the bell occupied by air at T°, y that occupied by air at $(T+t)^\circ$.

$$ah=x\,(h+d+x),\quad \text{Art. 99},$$
$$=y\,(h+d+y)\Big/\left(1+\frac{t}{T}\right).$$

The diminution of the tension of the chain is equal to $\frac{y-x}{a}W$, the difference of weight of water displaced.

Now $$2x=\{(h+d)^2+4ah\}^{\frac{1}{2}}-(h+d),$$

$$2y=\left\{(h+d)^2+4ah\left(1+\frac{t}{T}\right)\right\}^{\frac{1}{2}}-(h+d),$$

$$\therefore\ 2(y-x)=\{(h+d)^2+4ah\}^{-\frac{1}{2}}\frac{2aht}{T}\ \text{nearly}.$$

$\therefore$ the change of tension is

$$Wht/T\,\{(h+d)^2+4ah\}^{\frac{1}{2}}\ \text{nearly}.$$

12. Let a pressure p be reduced by one stroke of the piston to mp.

The acceleration after one stroke is $(1-m)g$,
after two strokes $(1-m^2)g$ etc.

If n strokes be made at equal intervals t, the velocity acquired at the instant of the nth stroke is

$$gt\left\{n-1-\frac{m(1-m^{n-1})}{1-m}\right\}$$
$$=gt\left\{n-\frac{1-m^n}{1-m}\right\}.$$

Just before the $(n+1)$th stroke, the velocity is

$$gt\left\{n-\frac{m(1-m^n)}{1-m}\right\}.$$

13. Here $m=\dfrac{n}{n+1}$ and the above velocity is

$$gnt\left\{\frac{n}{n+1}\right\}^n.$$

The limiting value of this, when n is infinite, is the value of

$$gnt\Big/\left(1+\frac{1}{n}\right)^n=\frac{gnt}{e}=\frac{v'}{e},\text{ the whole time, } nt\text{, being given.}$$

$$\therefore\ v'=ev.$$

14. Let x be the length of the fluid in the leg AB, $\therefore\ l-x$ the length in BC, a the angle ABC.

Since the free surface must pass through both ends of the fluid, its vertex being at the end in AB,

$$(l-x)^2\sin^2 a=\frac{2g}{\omega^2}\{(l-x)\cos a-x\}$$

is the equation giving x in terms of l.

We obtain

$$l-x=\frac{g}{2\omega^2\sin^2\frac{a}{2}}\left\{1\pm\sqrt{1-\frac{2\omega^2 l}{g}\tan^2\frac{a}{2}}\right\}.$$

If $$\omega^2>\frac{g}{2l}\cot^2\frac{a}{2},$$

this expression becomes imaginary.

Now the greatest value which $l-x$ can have is

$$2x\sec a.\quad \text{(See Art. 206 (2).)}$$

$\therefore$ if $$l>x(1+2\sec a),$$

some of the fluid will run out.

In the limiting case,

$$4x^2 \tan^2 a = \frac{2g}{\omega^2} . x ; \qquad \therefore\ x = \frac{g}{2\omega^2} \cot^2 a,$$

$$l = \frac{g}{2\omega^2} \cot a [\cot a + 2 \operatorname{cosec} a],$$

$$\frac{2\omega^2 l}{g} = \frac{\cos^2 a + 2 \cos a}{\sin^2 a} = \cot^2 \frac{a}{2} - \operatorname{cosec}^2 a.$$

$\therefore$ before ω reaches the value given, water begins to flow out of the tube and continues so to flow till all is gone.

15. Let x be the radius of the bubble at depth h.

Neglecting surface tension, we have

$$3c^3 = 2x^3 \quad \text{or} \quad x = c\sqrt[3]{3/2}.$$

If the surface tension be t, p, p' the internal pressures at radii c, x, w the intrinsic weight of water,

$$3wh - p = \frac{2t}{c},$$

$$2wh - p' = \frac{2t}{x}.$$

Also

$$pc^3 = p'x^3.$$

$$\therefore\ c^3 \left(3wh - \frac{2t}{c}\right) = x^3 \left(2wh - \frac{2t}{x}\right),$$

$$x^3 - \frac{t}{wh} x^2 + \frac{t}{wh} c^2 - \frac{3}{2} c^3 = 0$$

is the equation which determines x.

16. Employing the notation of Art. 99, the work done in depressing the water surface through a small distance d

$$= w(h + a + x) Ad.$$

$\therefore$ the whole work done in depressing the water surface from x' to x''

$$= \Sigma w (h + a + x) Ad = w\bar{z}A,$$

if $\bar{z}$ is the depth of the c. g. of the water displaced below the horizontal plane at the height h above the level of the water outside.

17. Let $a \cos a$ be the height of the free surface of the fluid at rest above the surface, a being the radius.

The volume of the rest of the sphere is

$$\frac{\pi a^3}{3} (1 - \cos a)^2 (2 + \cos a).$$

If the rotating liquid rise to a height $a\cos\theta$ above the centre, the free space is

$$\frac{\pi a^3}{3}(1-\cos\theta)^2(2+\cos\theta)+\pi a^2\sin^2\theta\,.\,\frac{\omega^2a^2\sin^2\theta}{4g},$$

$$\therefore\ \cos^3\alpha-3\cos\alpha-\cos^3\theta+3\cos\theta=\frac{3\omega^2a}{4g}\sin^4\theta.$$

The greatest elevation is $a(\cos\theta-\cos\alpha)$.

The greatest depression is

$$\frac{\omega^2a^2\sin^2\theta}{2g}-a(\cos\theta-\cos\alpha).$$

The latter is greater than the former if

$$\frac{\omega^2a}{4g}\sin^2\theta>\cos\theta-\cos\alpha.$$

Now

$$\frac{\omega^2a}{4g}\sin^2\theta/(\cos\theta-\cos\alpha)=\frac{1}{3}\,\frac{3-(\cos^2\theta+\cos\alpha\cos\theta+\cos^2\alpha)}{\sin^2\theta}.$$

$\therefore$ the above inequality holds if $2\cos^2\theta-\cos^2\alpha-\cos\alpha\cos\theta$ is positive.

But $$\cos\theta>\cos\alpha.$$

$\therefore$ this is true.

$\therefore$ the greatest depression exceeds the greatest elevation.

18. Let h be the height above the base of the cylinder of the top of the solid at first, and k when there is equilibrium.

Let z be the greatest height above the top of the solid of the surface of the liquid above it, and x the depth below the top of the solid of the paraboloidal surface continued.

Then $$\pi a^2h-\tfrac{1}{2}\pi a^2\,.\,\frac{a^2\omega^2}{2g}=\text{volume of liquid}$$

$$=\pi a^2k-\tfrac{1}{2}\pi a^2\,.\,\frac{a^2\omega^2}{2g}+\text{volume of liquid above,}$$

$$\therefore\ \pi a^2(h-k)=\pi a^2z-\tfrac{1}{2}\pi a^2(z+x)+\tfrac{1}{2}\pi\frac{2g}{\omega^2}x\,.\,x,$$

but $$a^2=\frac{2g}{\omega^2}(z+x),$$

$$\therefore\ \pi a^2(h-k)=\pi a^2\left(\frac{a^2\omega^2}{4g}-x\right)+\frac{\pi gx^2}{\omega^2}.$$

Also $$\sigma\frac{\pi a^2}{2}\,.\,\frac{a^2\omega^2}{2g}=\rho\left\{\frac{\pi a^2}{2}\,\frac{a^2\omega^2}{2g}-\frac{\pi}{2}\,.\,\frac{2g}{\omega^2}x\,.\,x\right\},$$

$$\therefore\ x^2=\frac{\sigma-\rho}{\rho}\,\frac{a^4\omega^4}{4g^2}\ \text{ and }\ h-k=\frac{a^2\omega^2}{4g}\left\{1-\sqrt{\frac{\sigma-\rho}{\rho}}\right\}^2.$$

19. Let v be the volume of each sphere, ρ, ρ' their densities, σ that of the fluid.

The straight line joining the spheres must intersect the axis of rotation at right angles.

If a and b are the distances from the axis,

$$\sigma v\omega^2 a + T = \rho v\omega^2 a,$$
$$\sigma v\omega^2 b - T = \rho' v\omega^2 b;$$

so that, if l be the length of the string,

$$\sigma (a+b) = \rho a + \rho' b, \qquad a - b = l.$$

If we take account of gravity the line of the string must intersect the axis and may be inclined to it at an angle θ.

We then have the equations

$$\sigma v\omega^2 a + T\sin\theta = \rho v\omega^2 a,$$
$$\sigma v\omega^2 b - T\sin\theta = \rho' v\omega^2 b,$$
$$g\sigma v + T\cos\theta = g\rho v,$$
$$g\sigma v - T\cos\theta = g\rho' v.$$

20. If the weight of the piston is equal to that of a depth y of water, and if x is the depth of the aperture below the piston, the jet rises to the height $x+y$.

21. 5000 cubic feet of water are raised 12 feet and heated through 6°.

The work done in four hours, in foot-pounds,

$$= 60000 \times 62{\cdot}5 + 30000 \times 62{\cdot}5 \times 772$$
$$= 1451250000.$$

Doubling this to obtain the actual work done by the engine, and dividing by 240×33000, we find that the H.P. is very nearly 366.

22. A repetition of Ex. 6, Chapter XV.

23. If water falls from a height h its velocity

$$v = \sqrt{2gh}.$$

If κ is the section of the tube, the volume of water which has its momentum destroyed in the time t is $v\kappa t$, and if m is the mass of unit volume, the momentum destroyed per unit of time $= mv^2\kappa$.

This is the pressure, in poundals, exerted by the falling water, and therefore if it support a column of the height x,

$$mv^2\kappa = m\kappa xg, \text{ and } \therefore x = 2h.$$

24. If f is the acceleration, downwards, of the bucket

$$f = \frac{mg}{2M+m}.$$

Let v be the volume of the cork, σ its density, so that $m = \sigma v$, and T the tension of the string.

If P is the upward pressure of the water, taking unity as the density of water,

$$vf=vg-P, \therefore P=\frac{2Mvg}{2M+m}.$$

Now
$$mf=mg-P+T,$$
$$\therefore T=\frac{2Mmg}{2M+m}\left(\frac{1}{\sigma}-1\right).$$

If V is the volume of water in the bucket, and h its height, $V=\pi r^2 h$.

Pressure on curved surface at first

$$=g\rho\pi r h^2=g\rho\frac{V^2}{\pi r^3}.$$

Afterwards, pressure $=(g-f)\rho\dfrac{(V+v)^2}{\pi r^3}$,

and this is greater or less than before according as

$$\frac{2M}{2M+m}(V+v)^2 > \text{ or } < V^2,$$

or
$$\frac{v}{V} > \text{ or } < \sqrt{1+\frac{m}{2M}}-1.$$

25. The acceleration of the bucket containing m is $\dfrac{m-m'}{2M+m+m'}g$ downwards.

Let T be the tension of the string, P the upward pressure of the water on m.

$$\frac{mg}{\sigma}-P=\frac{m}{\sigma}\frac{m-m'}{2M+m+m'}g,$$

$$T+mg-P=\frac{m(m-m')}{2M+m+m'}g.$$

$$\therefore T+mg\left(1-\frac{1}{\sigma}\right)=\frac{m(m-m')}{2M+m+m'}g\left(1-\frac{1}{\sigma}\right),$$

or
$$T=\frac{2m(M+m')g}{2M+m+m'}\left(\frac{1}{\sigma}-1\right).$$

26. If a be the area of the jet, v the velocity of efflux, x the depth.

In a short time τ a mass $\rho a v\tau$ emerges with velocity v.

The work done is $p\,.\,a\,.\,v\tau$.

$$\therefore pa\,.\,v\tau=\tfrac{1}{2}\rho a v^3\tau, \qquad \therefore v^2=\frac{2p}{\rho}=2gx.$$

The mass ρkvt emerging

$$=\rho kt\sqrt{\frac{2p}{\rho}}=kt\sqrt{2p\rho}.$$

The momentum emerging $=\rho kv^2t=2pkt$.

27. The spherical surface inside being smooth, it follows that the liquid revolves about the axis without rotation, and that if c is the distance of its centre from the axis, the acceleration of every element is the same and is equal to $\omega^2 c$ in the direction of that distance.

The resultant liquid pressure on any element m, combined with the force mg, produces the force $m\omega^2 c$.

The surfaces of equal pressure at any instant are parallel planes inclined to the vertical at the angle $\tan^{-1}(g/\omega^2 c)$.

28. As in the previous case the liquid revolves as a rigid body, without rotation, and if c is the distance CO of the axis of the aperture from the axis of the solid cylinder, the acceleration of every element is $\omega^2 c$ in the direction of that distance.

If at any instant θ be the inclination of CO to the vertical, and ϕ the inclination to the vertical of the resultant fluid pressure R upon an element m of the fluid,

$$R \sin\phi = m\omega^2 c \sin\theta, \qquad R\cos\phi - mg = m\omega^2 c\cos\theta,$$

$$\therefore\ \tan\phi = \omega^2 c \sin\theta/(g + \omega^2 c\cos\theta).$$

The surfaces of equal pressure are, at the instant, parallel planes inclined at the angle ϕ to the horizontal.

29. The quantity m is the ratio of $\omega^2 a$ to μ/a^2;

$$\therefore\ m\mu = \omega^2 a^3.$$

Taking a point near the equator let x be its distance from the axis, x being nearly equal to a, and let θ and ϕ be the small inclinations to the equator of the radius vector and the normal.

The attraction to the centre and the pressure in the normal have a resultant perpendicular to the axis, so that

$$\omega^2 x : \mu/r^2 = \sin(\phi - \theta) : \sin\theta,$$

$$\therefore\ \frac{\theta}{\phi} = 1 - \frac{\omega^2 a^3}{\mu} = 1 - m$$

and radius of curvature $=$ Limit of $\dfrac{r\sin\theta}{\phi} = a(1-m)$.

Again, taking a point P near the axis at the small distance x, let θ and ϕ be the small inclinations to the axis of the radius vector and the normal.

Then if the normal at P meet the equator in K, we obtain, from the triangle of forces CKP,

$$\omega^2 x : \mu/r^2 = \sin(\theta - \phi) : \cos\phi,$$

$$\therefore\ \theta - \phi = \frac{\omega^2 x b^2}{\mu} = \frac{m x b^2}{a^3}.$$

Now, if ρ is the radius of curvature,

$$\frac{\theta}{\phi} = \frac{\sin\theta}{\sin\phi} = \frac{\rho}{r} = \frac{\rho}{b}, \text{ and } x = \rho\phi,$$

$$\therefore\ \rho - b = \rho m b^3/a^3, \text{ or } b = \rho\{1 - mb^3/a^3\}.$$

30. If v is the velocity of efflux, $Ku=\kappa v$.

The equation of energy is that

$$\tfrac{1}{2}\rho(h-x)Ku^2+\tfrac{1}{2}\rho xKv^2+g\rho(h-x)K.\tfrac{1}{2}(h-x)$$

is constant, and is equal to $\frac{1}{2}g\rho h^2K$; so that

$$u^2\{(h-x)\kappa^2+xK^2\}=g\kappa^2(2hx-x^2).$$

This supposes that the issuing fluid enters into a horizontal tube fitting the aperture.

If however the fluid issues into the open air and is scattered, then the pressure at the aperture is that due to a height $h-x+u^2/2g$ of water.

If v be the velocity of efflux

$$v^2=2g(h-x)+u^2.$$

$$\therefore\ K^2u^2=\kappa^2\{2g(h-x)+u^2\},$$

$$u^2(K^2+\kappa^2)=2g\kappa^2(h-x).$$

31. The air in the jar can vibrate freely in the same period as the tuning-fork in the first case, but not in the other cases. It is therefore set in vibration, giving out the note natural to a jar of that depth in the first case, but being unable to vibrate in the same period as the fork in the other cases, gives but a slight sound.

32. The sound of the clapping is reflected from each rail and the series following rapidly one on the other and falling on the ear produces a sound resembling that produced by a cause which sets the air in vibration along a definite line, not instantaneously, but in rapid succession.

33. Consider the water above a plane touching the lowest point of the sphere in the first case.

Its volume is $\pi R^2.2r-\frac{4}{3}\pi r^3=2\pi r[R^2-\frac{2}{3}r^2]$.

Its c.g. is at a height r.

When the sphere is gone the height of the c.g. is

$$\frac{\pi r[R^2-\frac{2}{3}r^2]}{\pi R^2},$$

$\therefore$ it has fallen a distance $\dfrac{2}{3}\dfrac{r^3}{R^2}$,

$\therefore$ the loss of potential energy

$$=\frac{W}{\frac{4}{3}\pi r^3}.2\pi r\left[R^2-\frac{2}{3}r^2\right]\times\frac{2r^3}{3R^2}$$

$$=Wr[3R^2-2r^2]\div 3R^2.$$

When the sphere is half out of the water, the height of the water is

$$\frac{2r}{3}\frac{3R^2-r^2}{R^2}.$$

The height of its c.g. is

$$\frac{\frac{2\pi r}{3}(3R^2-r^2)\cdot\frac{r}{3}\frac{3R^2-r^2}{R^2}-\frac{2}{3}\pi r^3\cdot\left(\frac{2r}{3}\cdot\frac{3R^2-r^2}{R^2}-\frac{3r}{8}\right)}{\frac{2\pi r}{3}(3R^2-2r^2)},$$

$\therefore$ it has fallen $\frac{r^3}{8R^2}\frac{13R^2-8r^2}{3R^2-2r^2}$.

Loss of potential energy

$$=\frac{W}{2R^2}\frac{r^3}{8R^3}(13R^2-8r^2)=\frac{39R^2-24r^2}{48R^2-32r^2}\times\text{former loss.}$$

When the sphere leaves the water, w being its weight, it has gained in potential energy

$$w\cdot\frac{2}{3}\frac{r^3}{R^2},$$

$\therefore$ its k.e. must be

$$\frac{Wr}{3R^2}(3R^2-2r^2)-w\cdot\frac{2r^3}{3R^2}.$$

Its velocity is therefore

$$\sqrt{\frac{gr}{3R^2}\left(\frac{W}{w}(3R^2-2r^2)-2r^2\right)}.$$

34. The time of vibration of the fifth of G is $\frac{2}{3}$ that of G and therefore $\frac{4}{9}$ that of C.

35. The number of strokes per second made by the teeth on the card is $4\times 264=1056$.

$\therefore$ it revolves $\frac{1056}{33}=32$ times per second.

36. Velocity of train $=88$ feet per second.

If n be the vibration number of the note sounded, that of the note heard as the train approaches is $\frac{1120}{1120-88}n=\frac{140}{129}n$.

The frequency of the note heard as the train recedes is

$$\frac{1120}{1120+88}n=\frac{140}{151}n.$$

If the note be the fifteenth of the middle C,

$$n=1056.$$

$\therefore$ the vibration numbers of the apparent notes are

$$1145\tfrac{35}{129}\text{ and }979\tfrac{11}{151}.$$

The first is sharpened slightly more than half a tone, the second flattened slightly less than half a tone.

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